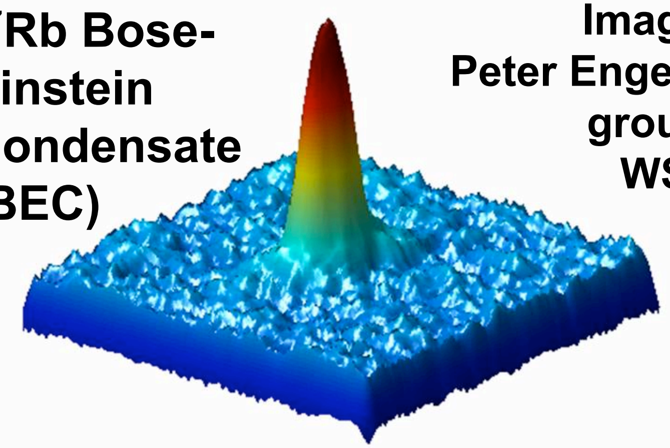

Few-Body Aspects of Bose and Fermi Gases

Theoretical frameworks for treating cold few-particle systems: Two-body scattering and hyperspherical perspective

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Collaborators:
At WSU: Gabriel Hanna, Krittika Kanjilal, M. Asad-uz-Zaman, Kevin Daily, Debraj Rakshit, Ryan Kalas, Kris Nyquist.
At JILA: Chris Greene, Javier von Stecher, Seth Rittenhouse, John Bohn, Shai Ronen, Danielle Bortolotti.
At BEC Center in Trento: Stefano Giorgini, Grigori Astrakharchik.

Supported by NSF and ARO.

Outline of This Talk

- Broad introduction:
  - Cold atom physics.
  - Few-body physics.
  - Overview of lectures.

  - Two-body interactions and scattering length.
  - Connection between microscopic world and mean-field formulation.
  - Microscopic understanding of stability of Bose and Fermi gases: Linear Schroedinger equation within hyperspherical framework.

The Field of Cold Atom Physics

EXPERIMENT ↔ THEORY

Nobel Prizes:

Laser cooling (1997):
Chu, Cohen-Tannoudji, Phillips.

Bose-Einstein condensation (2001):
Cornell, Ketterle, Wieman.

Theory of superconductors and superfluids (2003):
Abrikosov, Ginzberg, Leggett.

Quantum optics and frequency comb (2005): Glauber, Hall, Hansch.

molecular physics

nuclear physics

condensed matter

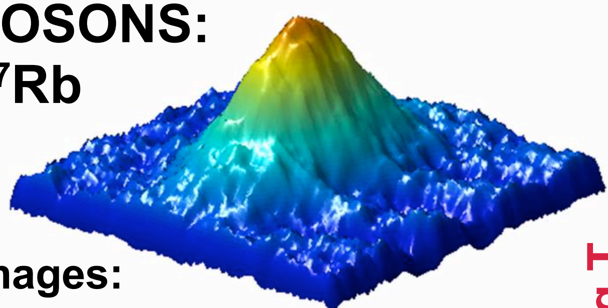
atomic physics

quantum information science

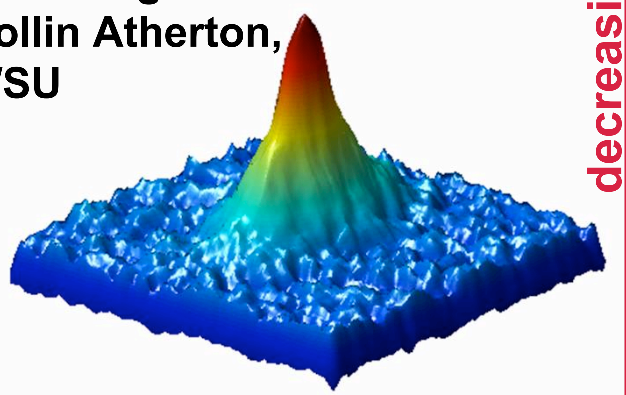
quantum optics

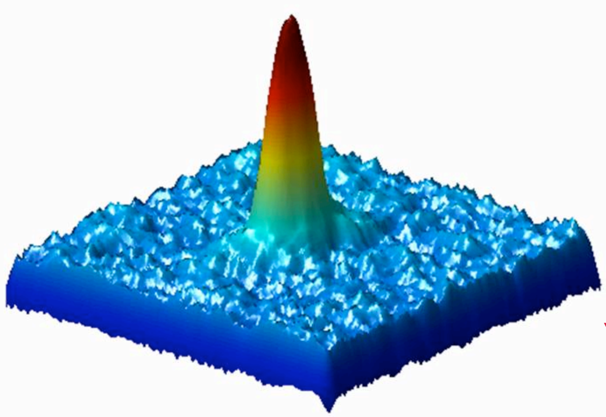
Reaching Quantum Degeneracy: Bosons versus Fermions

BOSONS:
^{87}Rb

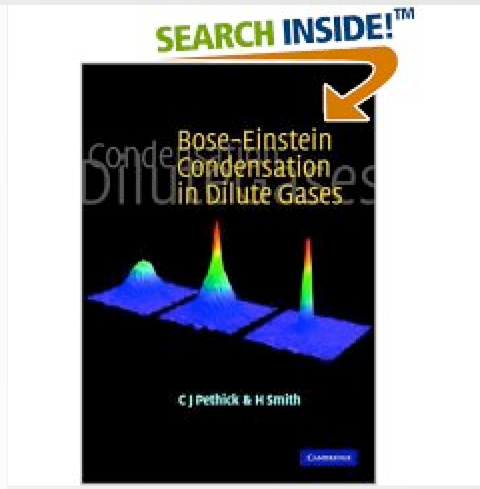


Images:
Peter Engels and
Collin Atherton,
WSU





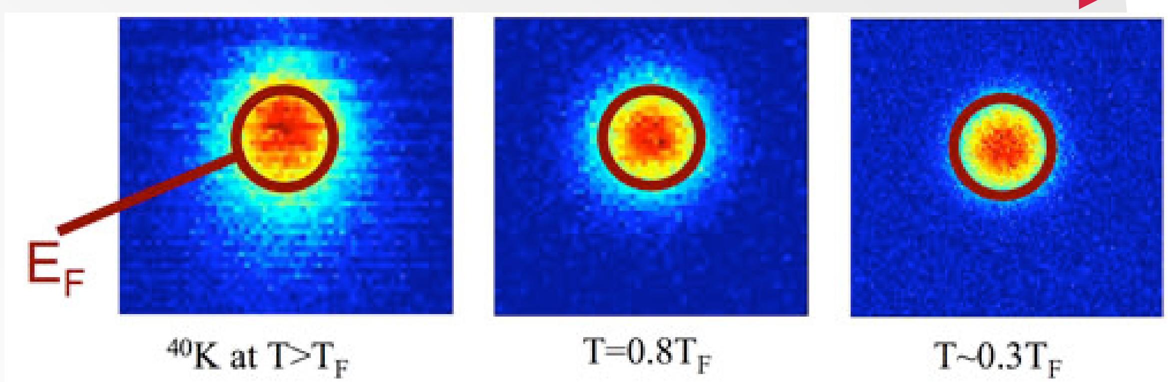
decreasing T



Transition temperature of bosons: a few 100nK (BEC: the coldest place in the known Universe)

FERMIONS:

decreasing T



Images: Thywissen group, University Toronto.

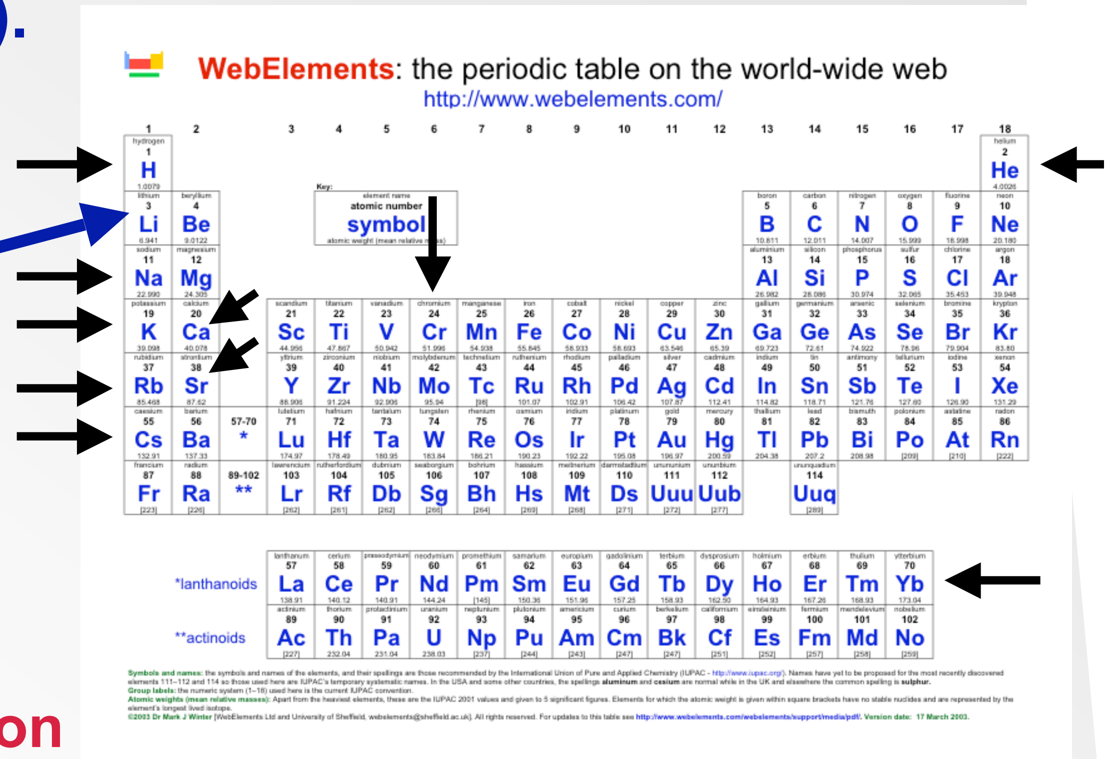
Composite Atomic Bosons versus Fermions.

- Boson: Integer spin; e.g., photon, mesons (q, anti-q).
- Fermion: Half-integer spin; e.g., electron, quarks, proton, neutron, baryons (q,q,q).
- **Atoms: Composite bosons and fermions.**
- E.g.: ^{4}He (B) and ^{3}He (F).

^{6}Li (I=1): 3 electrons
3 neutrons
3 protons

Composite fermion

In contrast:
^{7}Li (I=3/2) composite boson



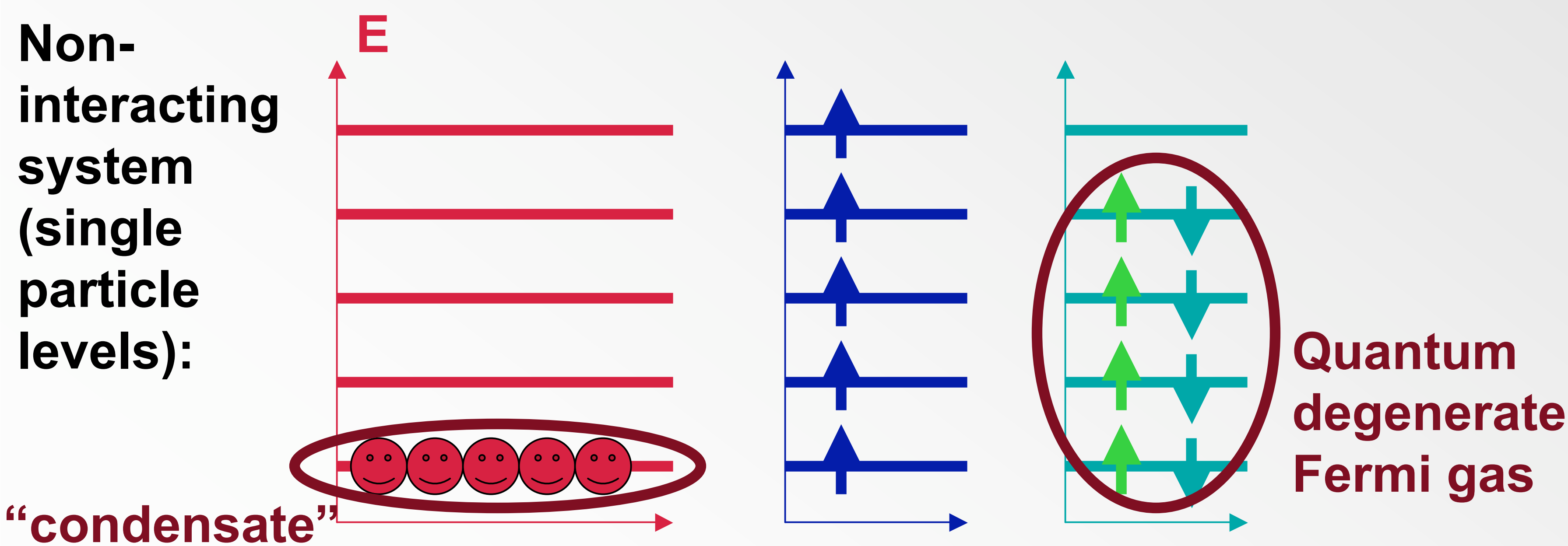
Bose versus Fermi Statistics: Gas at Low Temperature

One-component Bose gas:

One-component spin-polarized Fermi gas:

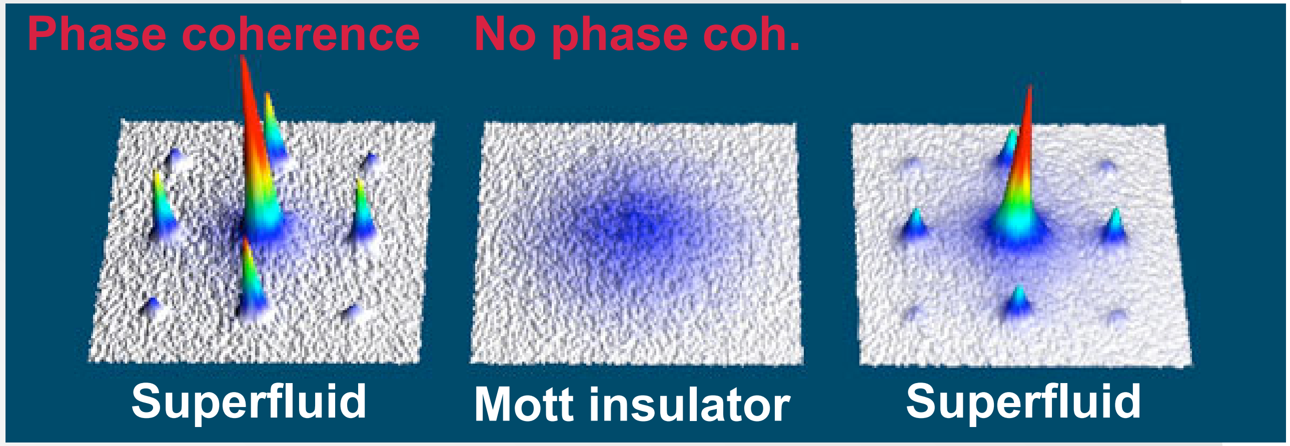
Two-component Fermi gas:

Non-interacting system (single particle levels):

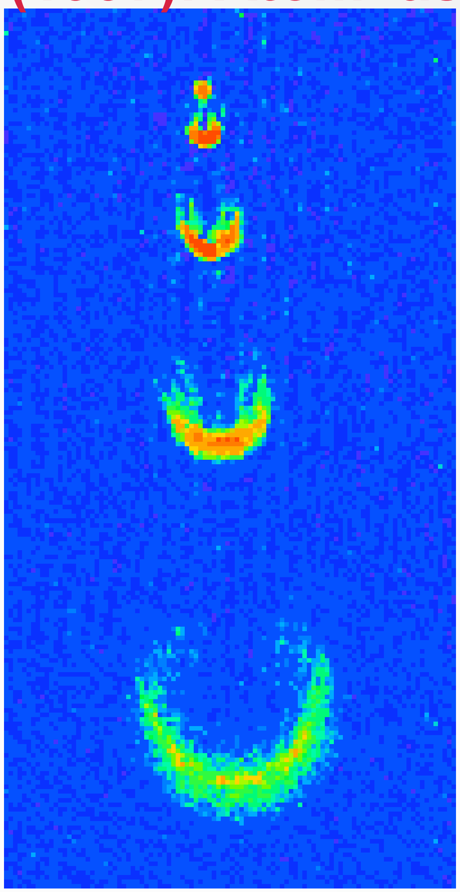


Selected Highlights from Cold Atom Experiments

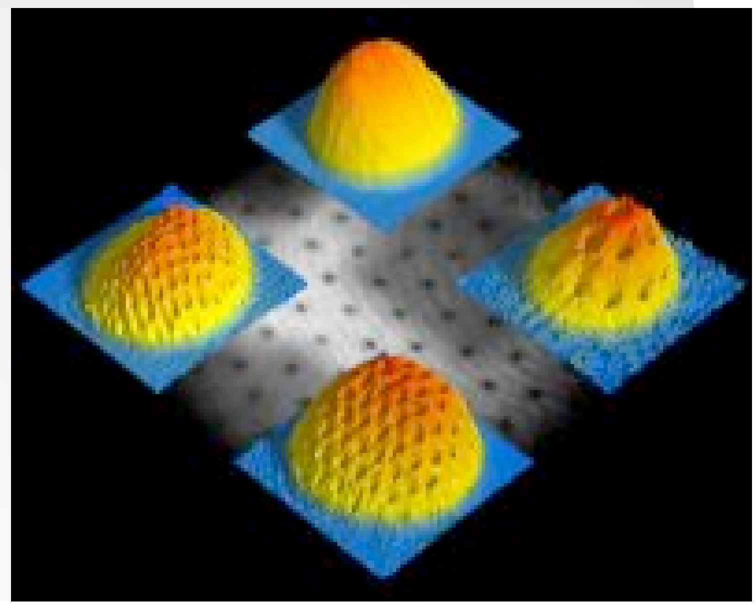
MPI, Hansch group (2002): Quantum phase transition:



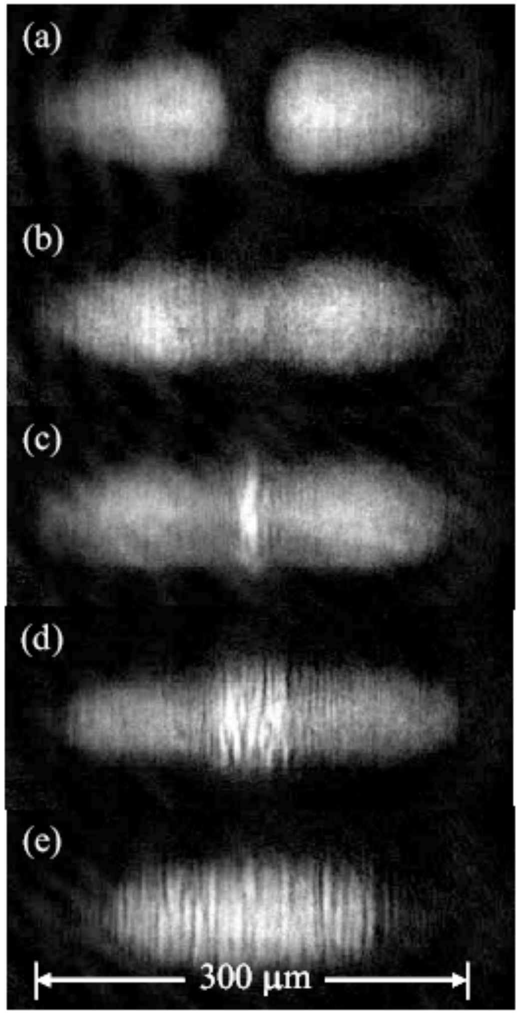
MIT, Ketterle group (1997): Atom laser.



MIT, Ketterle group (2001): Vortex lattice in Na BEC.

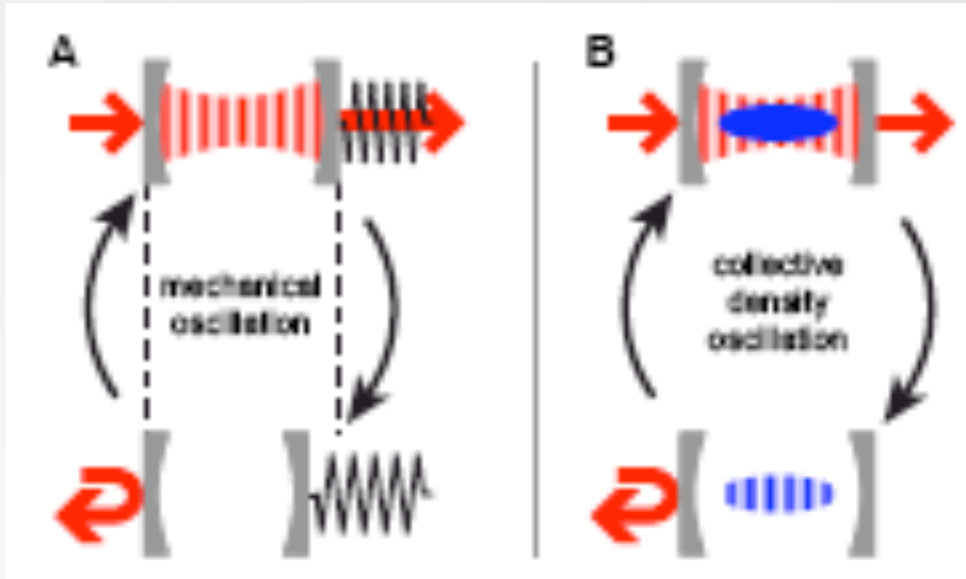


More Highlights from Cold Atom Experiments…

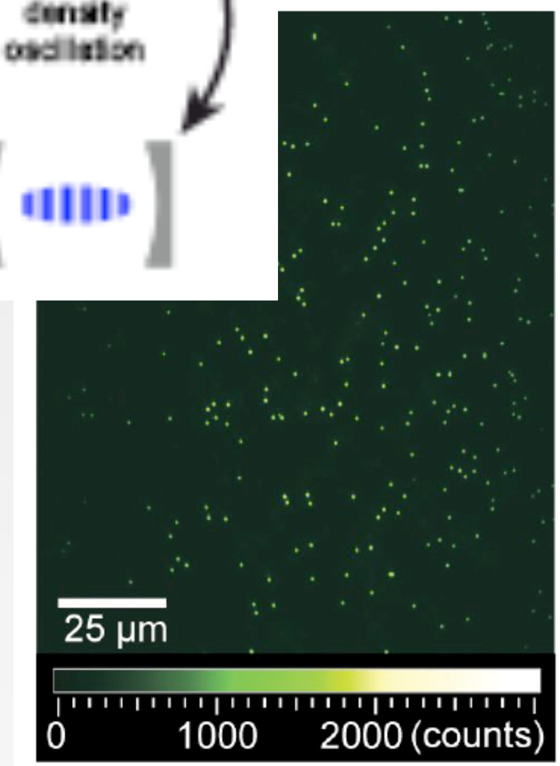


Quantum shock
Chang et al., PRL (2008)

Cavity opto-mechanics with a BEC
Brennecke et al., Science 322, 238 (2008)



Quantum gas microscope
Bakr et al., arXiv:0908.01744.



Few-Body Highlights: Optical Lattice

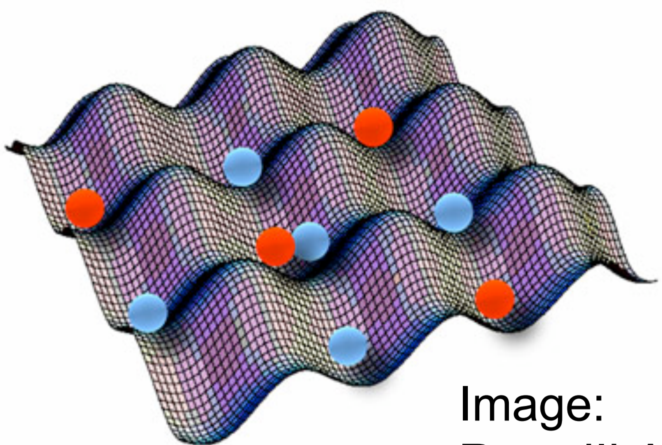
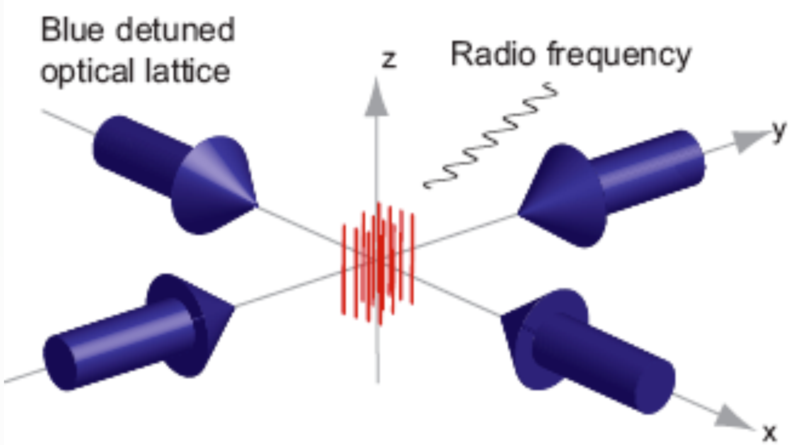


Image: Rey, JILA

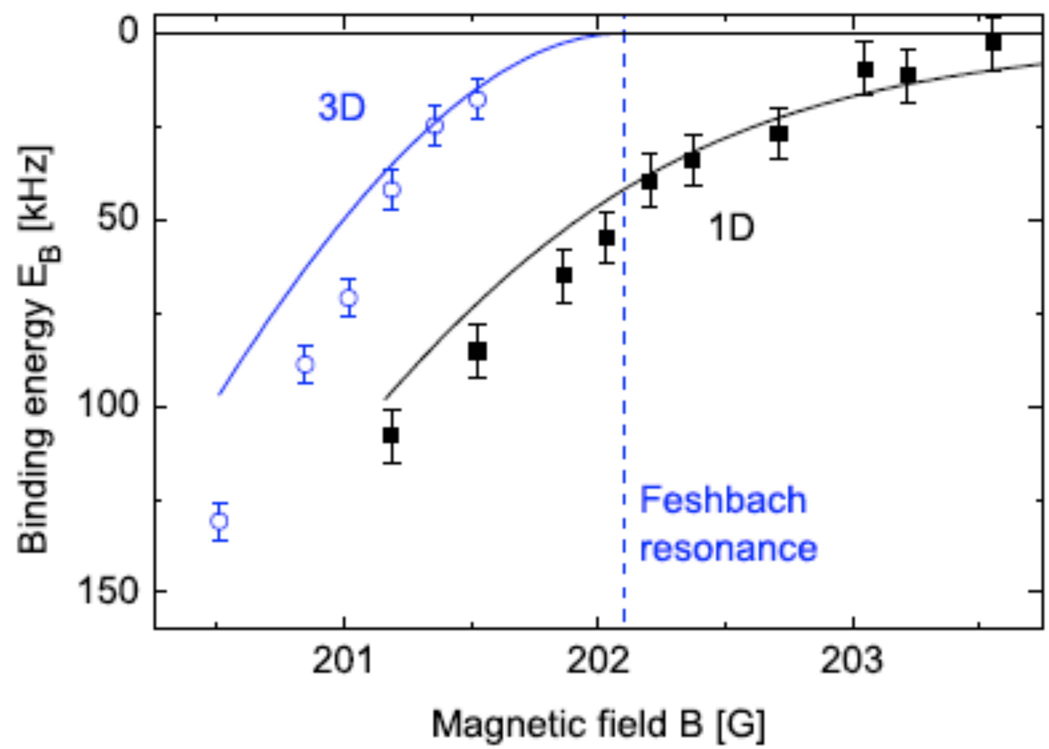
Designing effectively 1D and 2D confinement:



Palzer et al., PRL 103, 150601 (2009)

Measurement of two-body binding energy in 3D and 1D

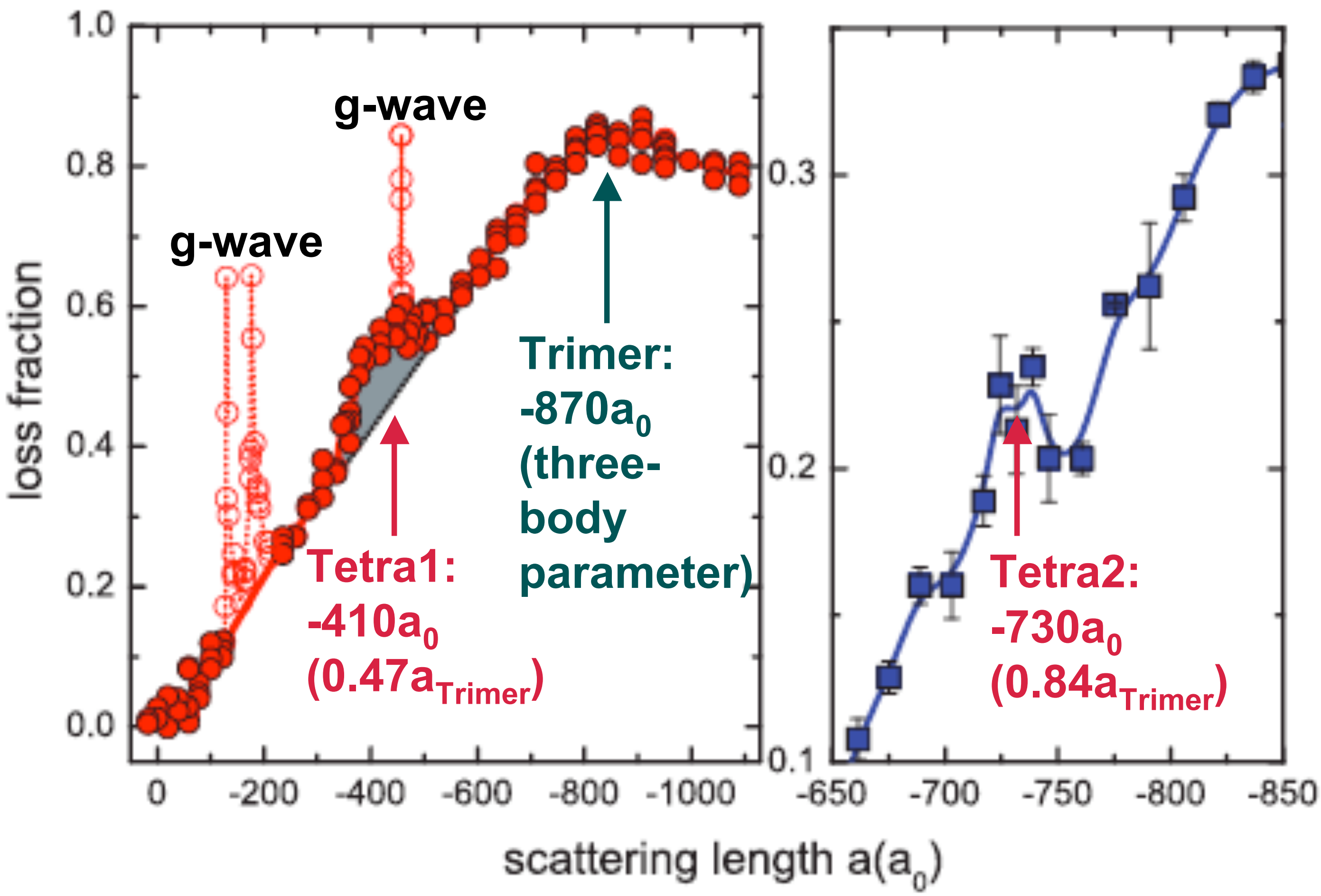
Moritz et al.,PRL 94, 210401 (2005)

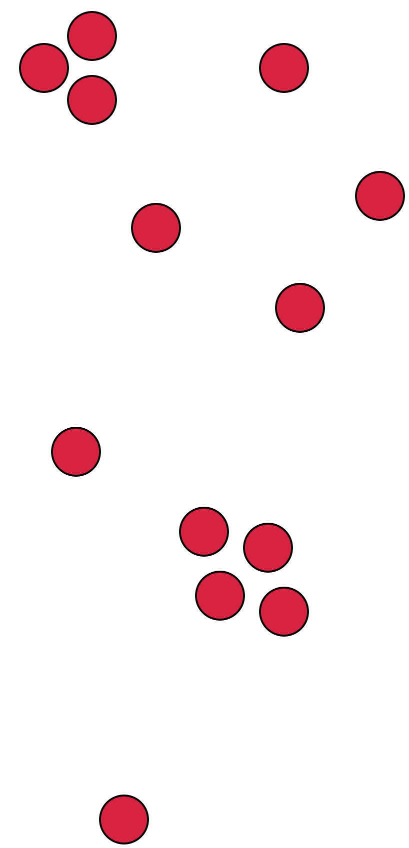


Measurement of Loss Rate for Non-Degenerate Bosonic ^{133}Cs Sample

First measurement of universal 4-body physics (probe of Efimov physics)

Ferlaino et al., PRL 102, 140401 (2009).

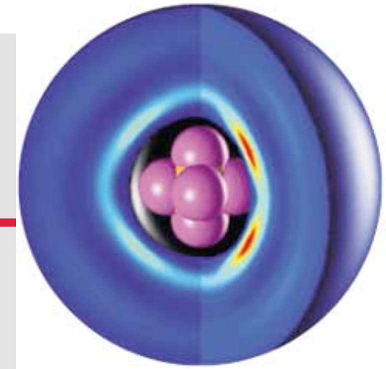




How to stabilize these delicate trimers and tetramers?

Few-Body Physics: Bridge to Macroscopic World

Toennies et al., Physics Today 54, 31 (2001).



- Microscopic to macroscopic:

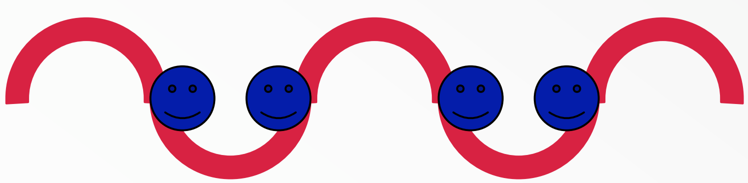
atomic/ molecular

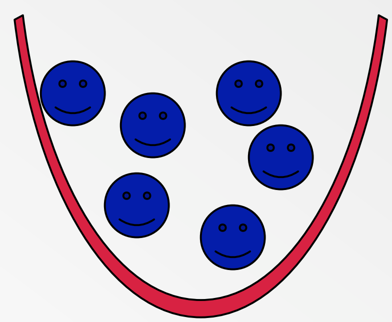
condensed matter

- Other examples:
  - Doped helium clusters: Molecular rotations, microscopic superfluidity,...
  - Metal clusters: conductivity, designing materials,...

- What is special about cold atomic Bose and Fermi systems?
  - Universal behavior.
  - Much experimental progress!

optical lattice





External confining potential

Topics to be Covered

equal-mass Fermi gas

unequal-mass Fermi gas

Bose gas

dipolar Bose gas

Scattering theory

Monte Carlo techniques

hyperspherical framework

Stochastic variational approach

Virial expansion

Mean-field theory

Generalized Scattering Length: Central Two-Body Interaction V(r)

- Two-particle system: Separate off CM degrees of freedom.

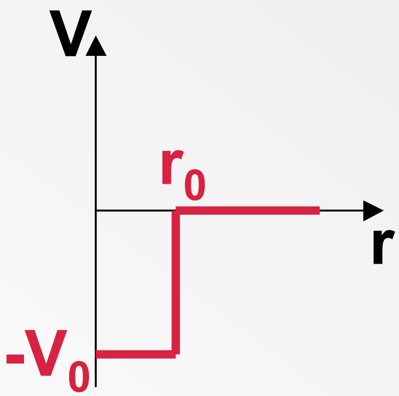
- Partial wave decomposition of wave function in relative coordinates: $\psi(r,\theta,\varphi) = \Sigma_{lm} R_l(kr)\, Y_{lm}(\theta,\varphi)$.
- Scaling: $u_l(kr) = kr\, R_l(kr)$.

- Outside: Radial solution defines phase shifts $\delta_l(k)$.
  $R_l(kr) = A_l(k)[j_l(kr) - \tan(\delta_l(k))\, n_l(kr)]$.

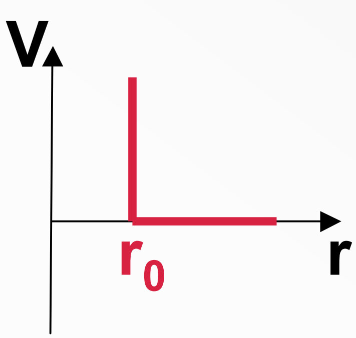
- Generalized energy-dependent scattering lengths (SR)
  $a_l(k) = -\tan(\delta_l(k)) / k^{2l+1}$.
- Generalized energy-independent scattering lengths
  $a_l = \lim_{k\to 0} a_l(k)$.

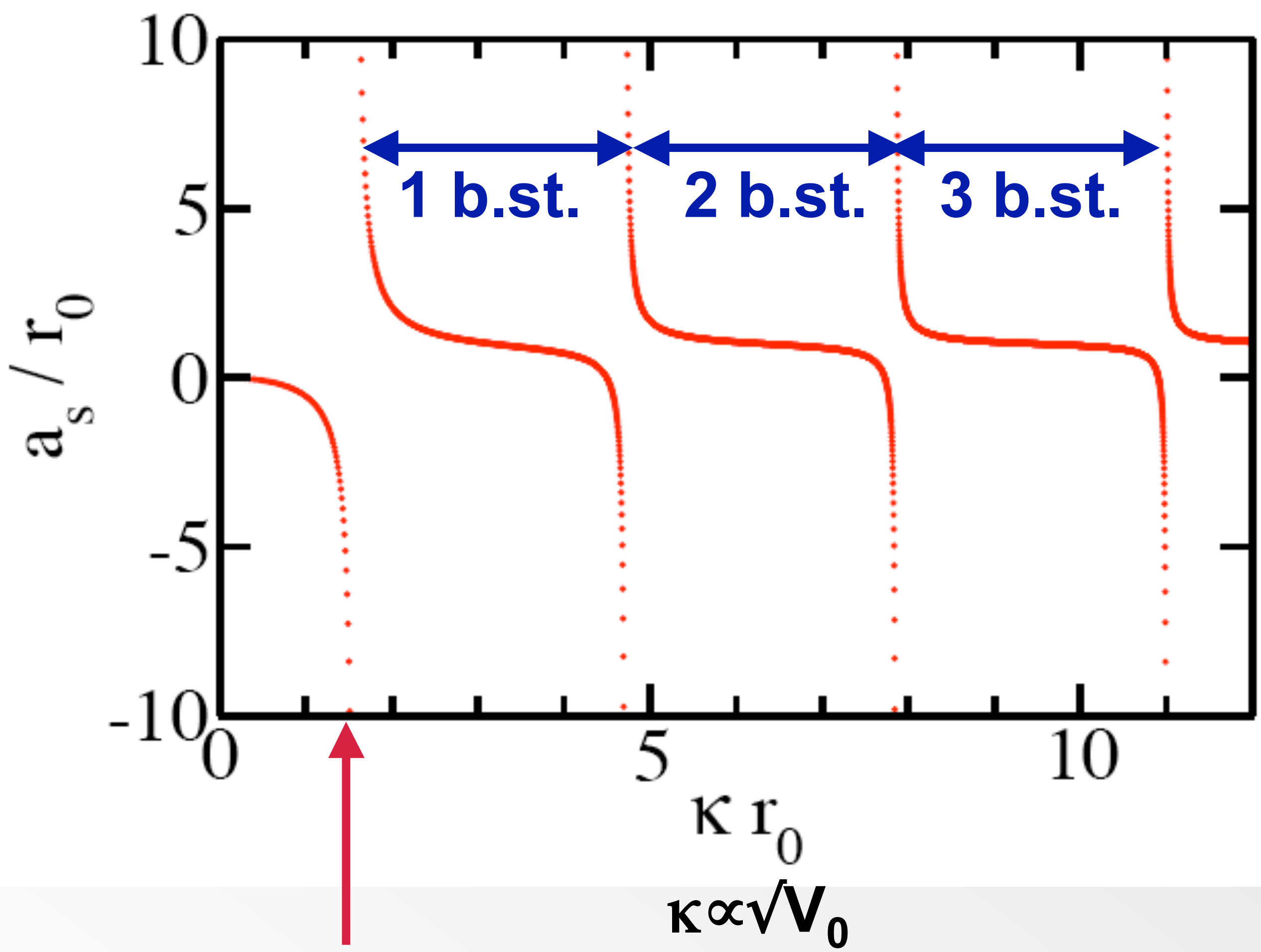
- Inside solution: Same as bound state solution.

Example:
Square-Well Interaction Potential



In contrast:
Hardcore potential:
$a_s = r_0$



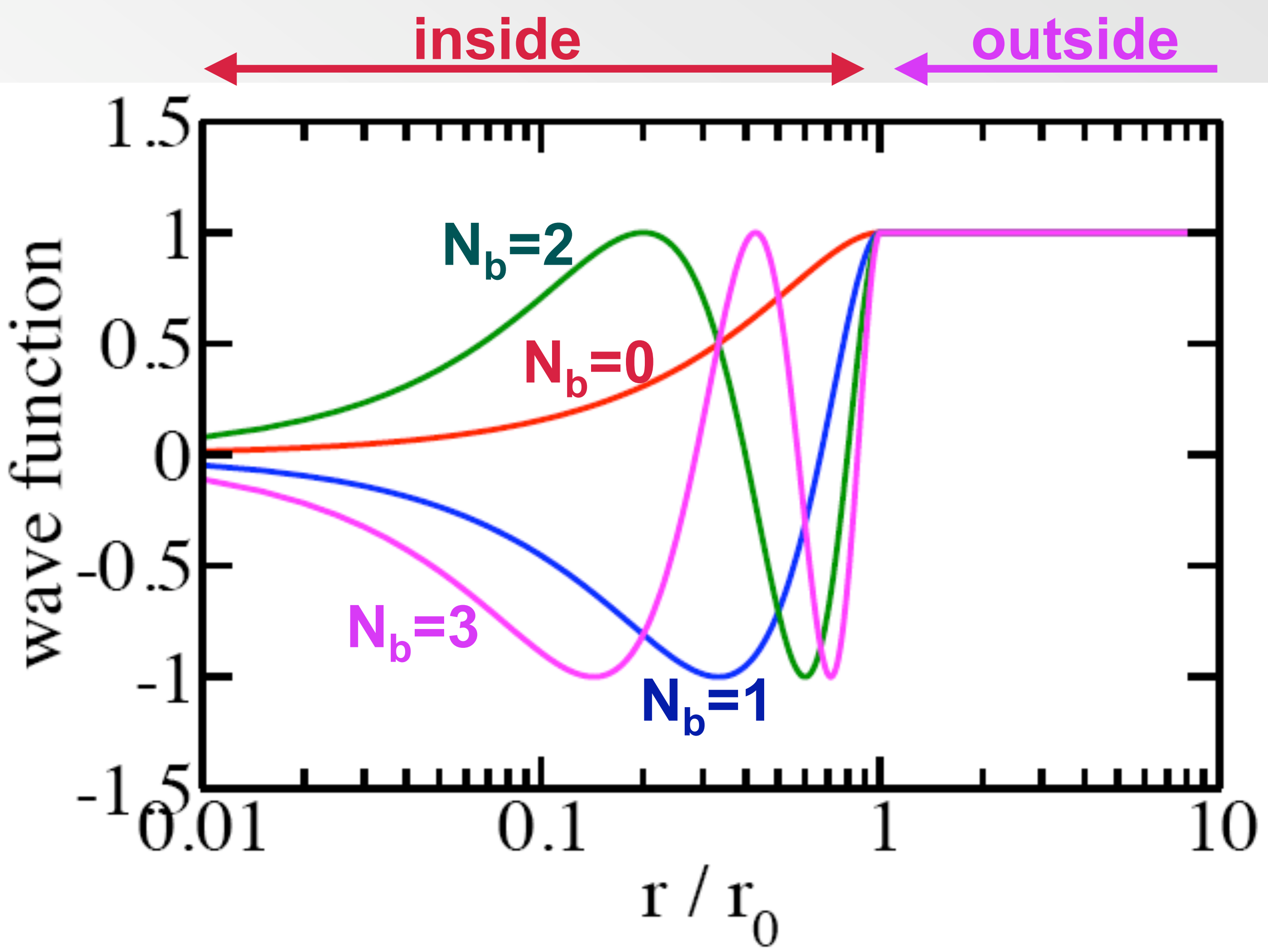


Infinite scattering length (unitarity). Small range.

$\kappa \propto \sqrt{V_0}$

increasing well depth (attraction)

Zero-Energy Scattering Wave Function at Unitarity



Outside solution identical for all four potential depths.

Outside ($a_s \rightarrow \infty$):

$u_s(r) \propto$

$\sin(kr) +$

$\tan(\delta_s(k))\cos(kr) \propto$

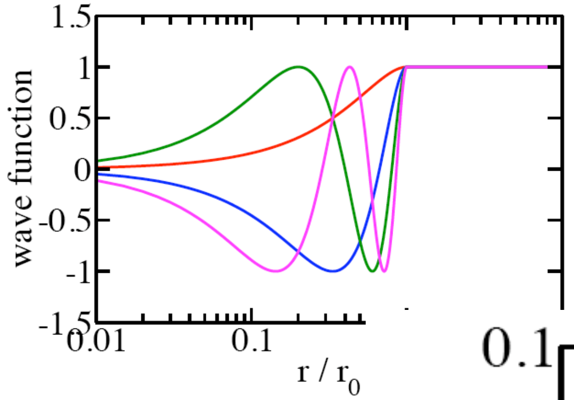
$kr + \tan(\delta_s(k)) \propto$

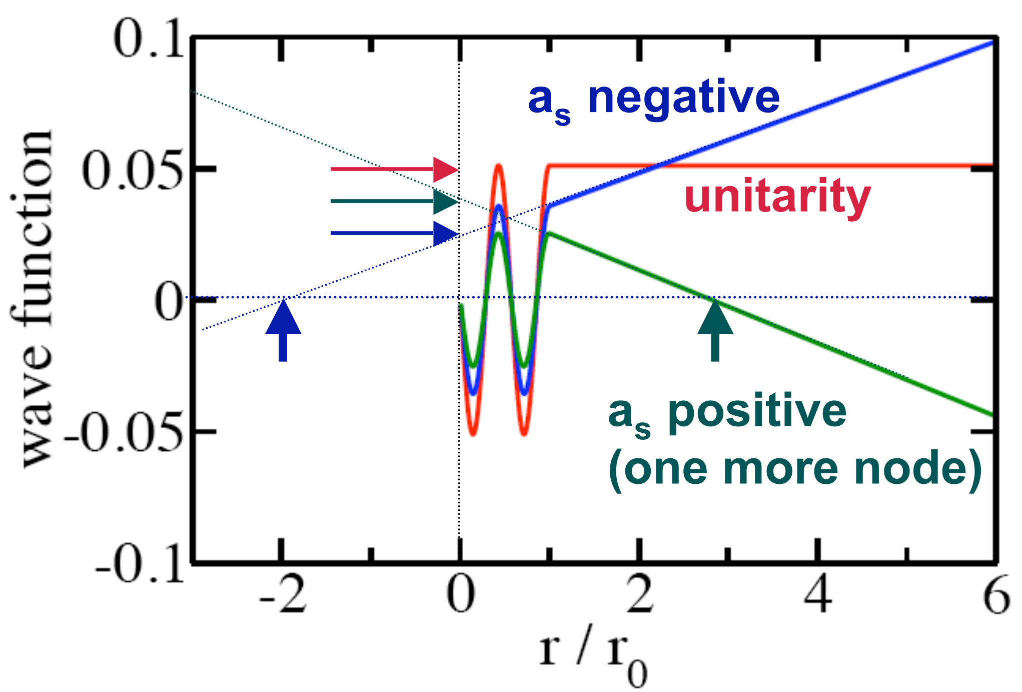
$r - a_s \propto r/a_s - 1 \propto$

const

Inside solution depends on details of interaction potential.

Zero-Energy Scattering Wave Function





Outside:

$u_s(r) \propto$

$\sin(kr)+$

$\tan(\delta_s(k))\cos(kr) \propto$

$kr + \tan(\delta_s(k)) \propto$

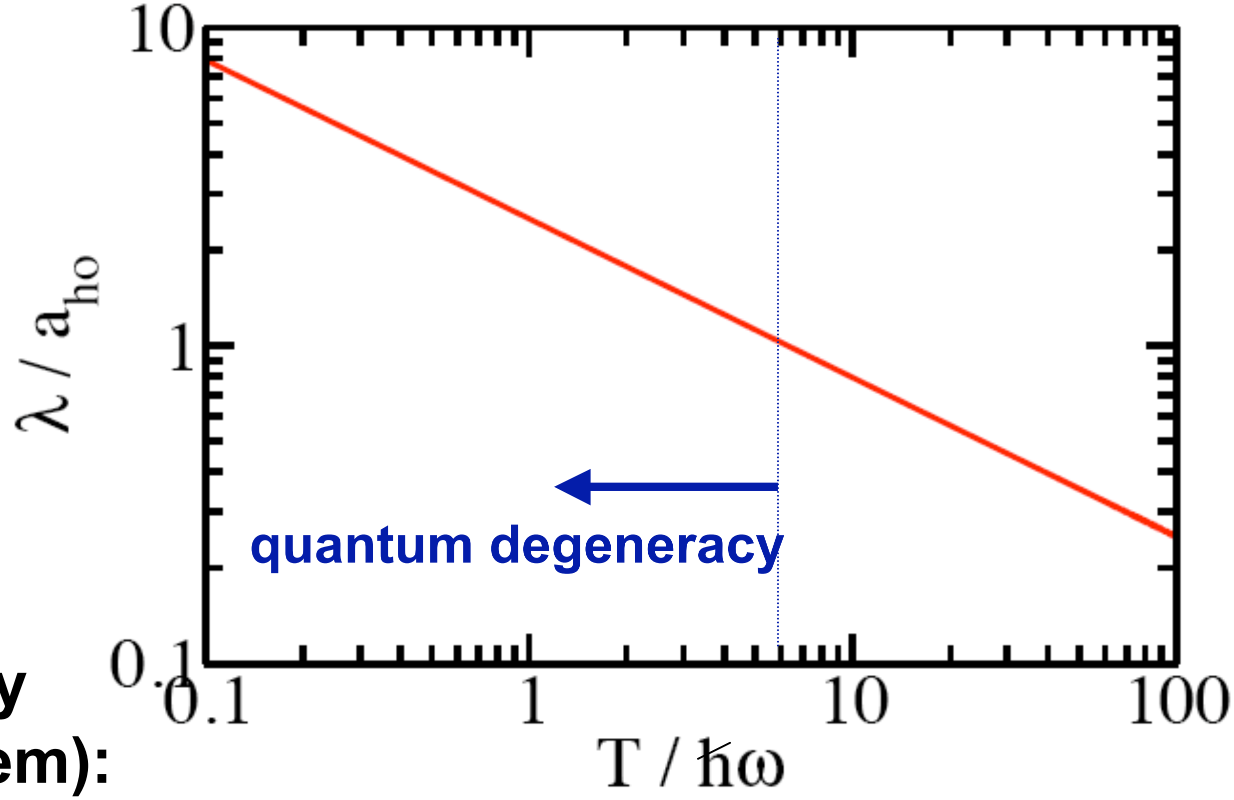
$r - a_s$

Inside solution depends on details of interaction potential.
These details are not being probed at low temperature because…

deBroglie Wave Length: Degeneracy and Resolution

$\lambda = h / (2\pi\, m\, k_B T)^{1/2}$

Assuming a trapped system with osc. length a_{ho}:
$\lambda / a_{ho} = (2\pi\, h\nu / k_B T)^{1/2}$



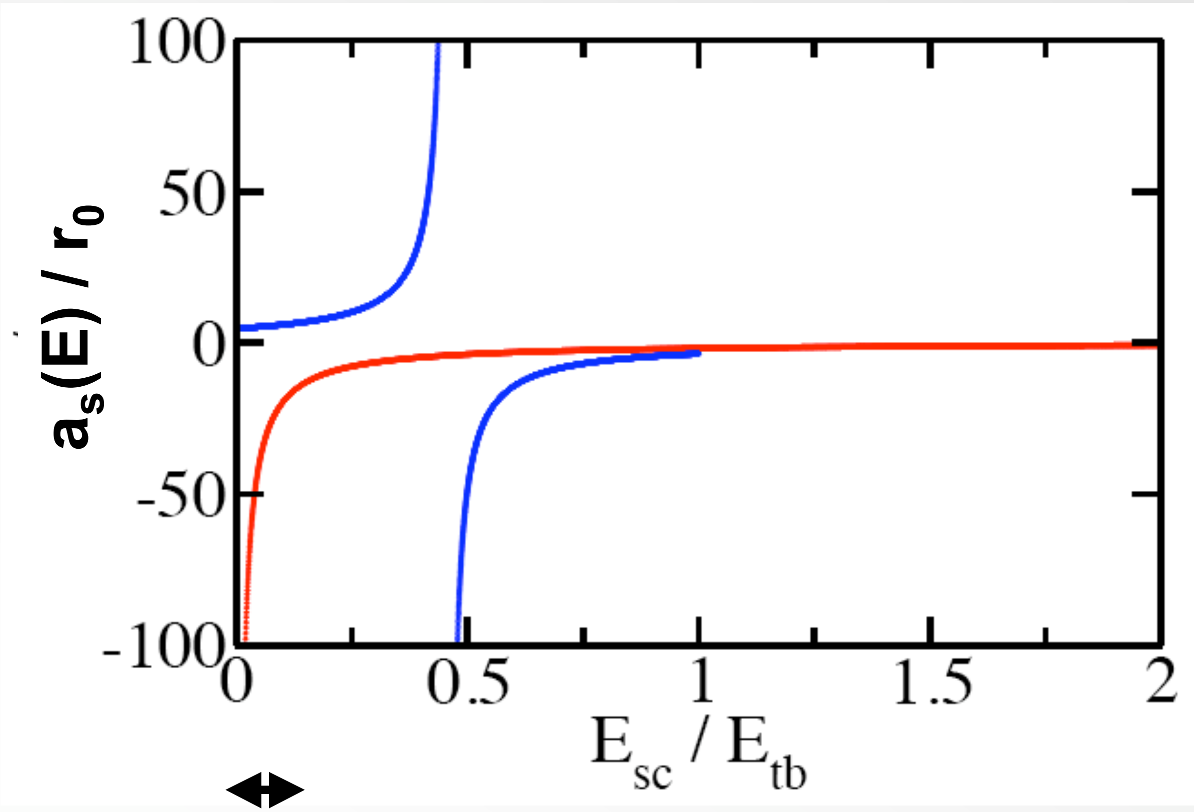
Quantum degeneracy (homogeneous system):
$\lambda / \langle r_{ij} \rangle > (2.6)^{1/3} \approx 1.38$

Resolution is set by λ ($r_0 << a_{ho}, \lambda$).

Low T: Collisions are too slow to probe small r piece (high momentum piece) of wave fct.

High T: Can probe small r piece of wave fct.

Temperature Determines Collision Energy



low E

$E_{tb} = \hbar^2/(2\mu r_0^2)$

Low energy physics implies:

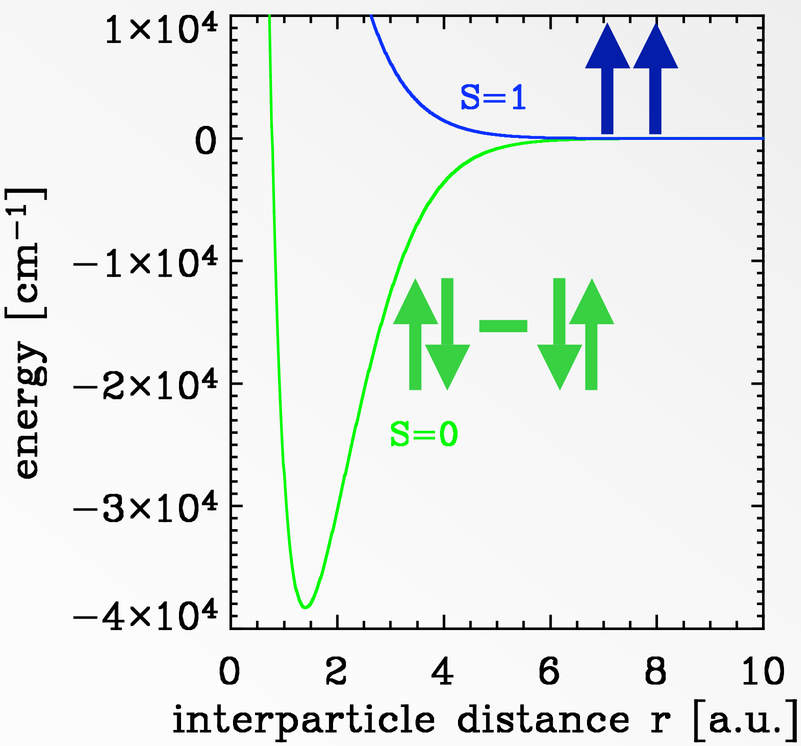
$E_{sc} << E_{tb}$

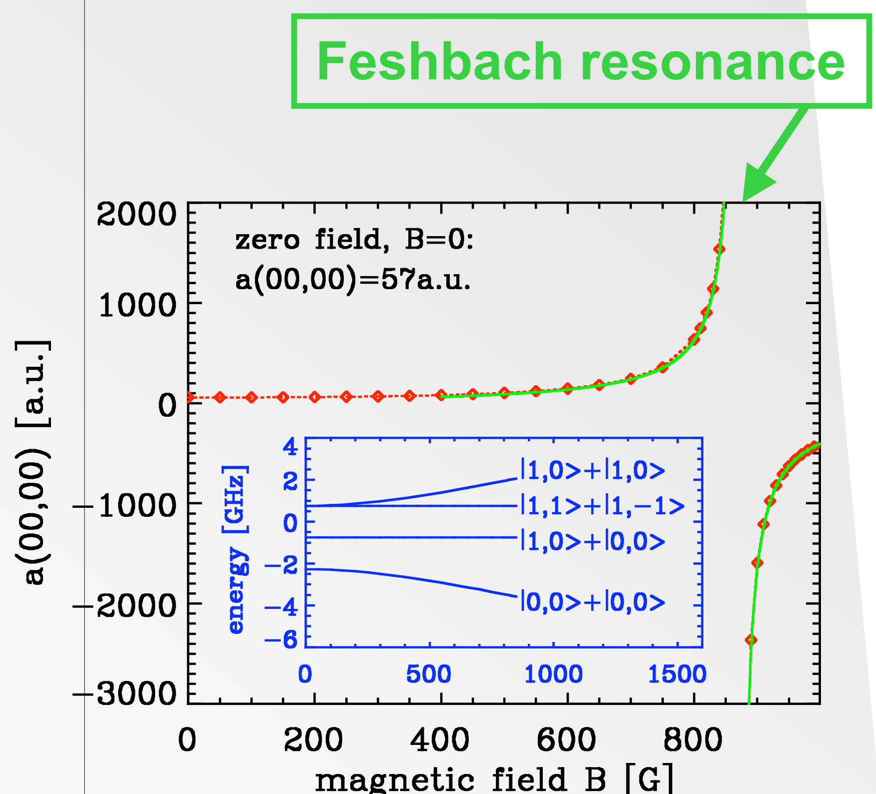
$E_{bound} << E_{tb}$
(dimer size larger than range r_0)

Weak binding:
$E_{bound} \approx -\hbar^2/ma_s^2$

But: Scattering Length Determined by Coupled Channel Interaction

- Hyperfine Hamiltonian couples singlet and triplet potential curves.
- In the vicinity of Fano-Feshbach resonance, scattering length tunable (here, tritium-tritium system).



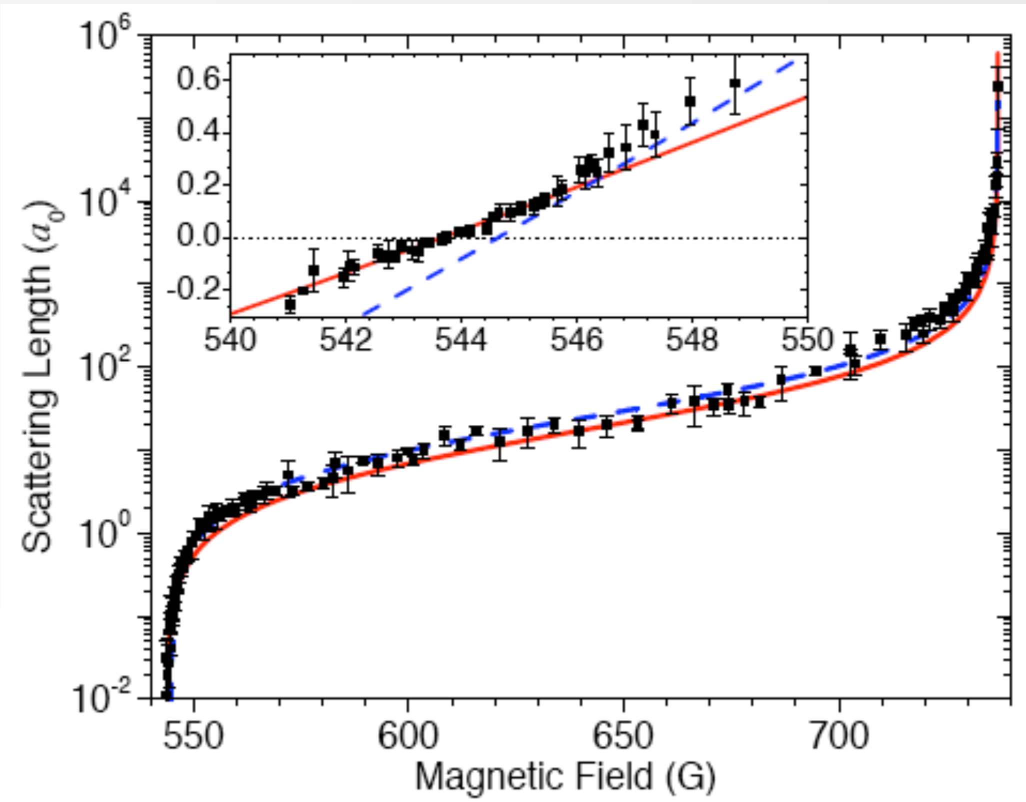


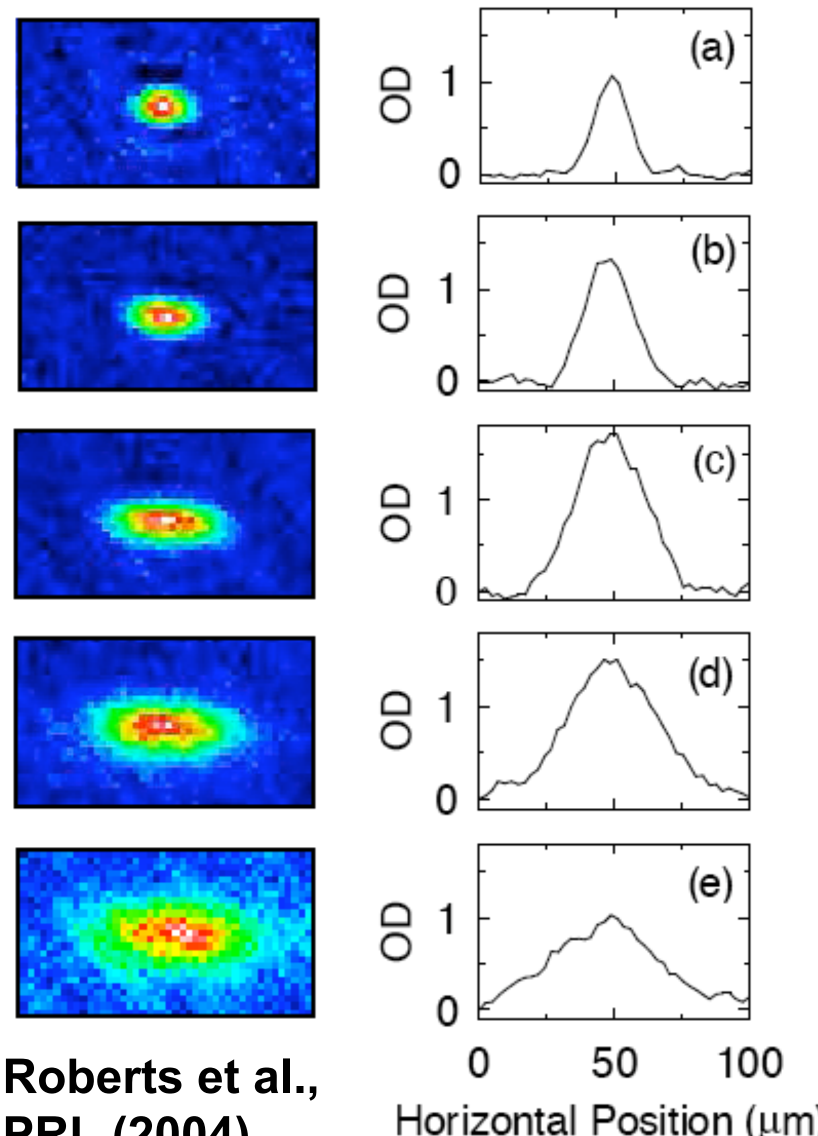
See Blume et al., PRL 89, 163402 (2002).

Experimental Feshbach Resonance Tuning

Extreme tunability of ^{7}Li Bose gas

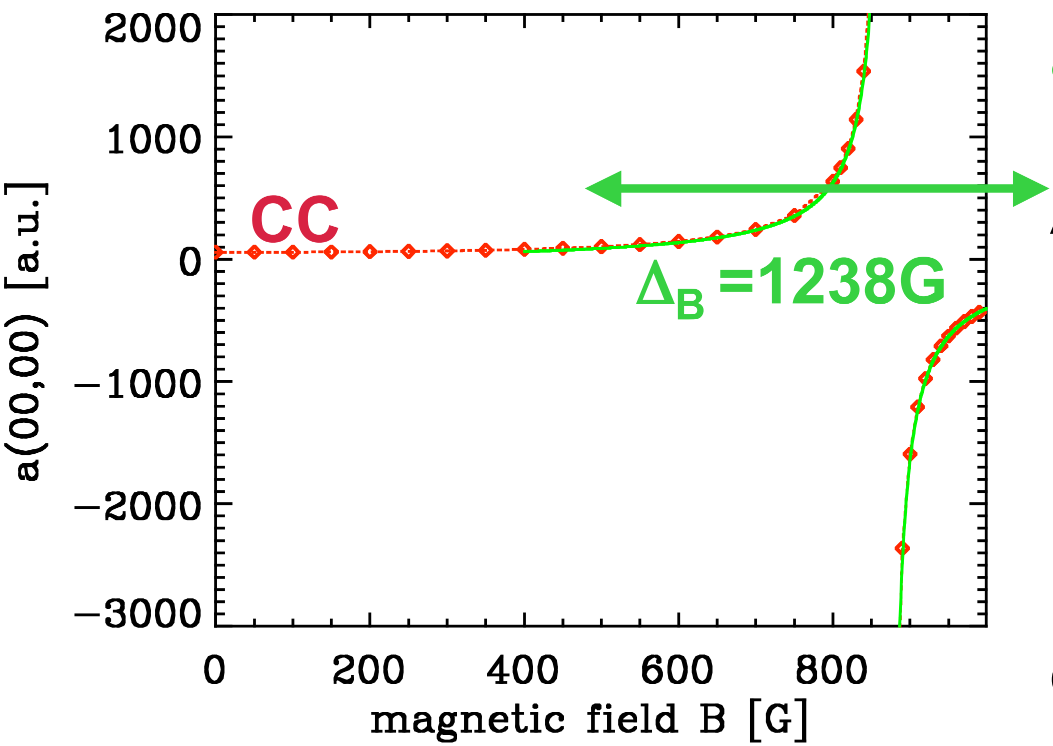
Pollack et al., PRL 102, 090402 (2009)





Roberts et al., PRL (2004)

Broad and Narrow Resonance: Can We Get Away With Single Channel Model?



$a_{eff} = a_{bg}(1+\Delta_B/(B-B_R))$

$\Delta_B \sim \Delta_E$

Broad resonance:

$E_{char} >> \Delta_E$ or $r^* k_{char} << 1$

$r^* = \hbar^2 / (m\, a_{bg}\, \Delta_B\, \mu_{oc})$

e.g., ^{40}K: $r^* \sim 10$Å

Fermi gas in strongly-interacting regime:

$k_{char} = k_F$ (negligible occupation of closed channel molecule)

Composite molecular Bose gas:

$E_{char} = E_{bind}$, $k_{char} = 1/a_s$

Physics Determined by s-Wave Scattering Length

- Reaching quantum degeneracy requires thermalization.
- Efficient thermalization requires finite a_s.
- No thermalization for single-component Fermi gas.

- Three examples:
  - 1. Gas-like states of trapped two-particle system are determined by s-wave scattering length a_s.
  - 2. Stability/instability of Bose gas determined by scattering length.
  - 3. Stability/instability of multi-component Fermi gas determined by scattering length plus Fermi pressure (Pauli exclusion principle).

1. Two-Particle System: Replace Atom-Atom Interaction by ZR Interaction

- Start with ab initio atom-atom potential.
- Coupled channel calculation provides phase shifts $\delta_l(k)$.

- Construct zero-range pseudo-potential with same a_s (outside solution):

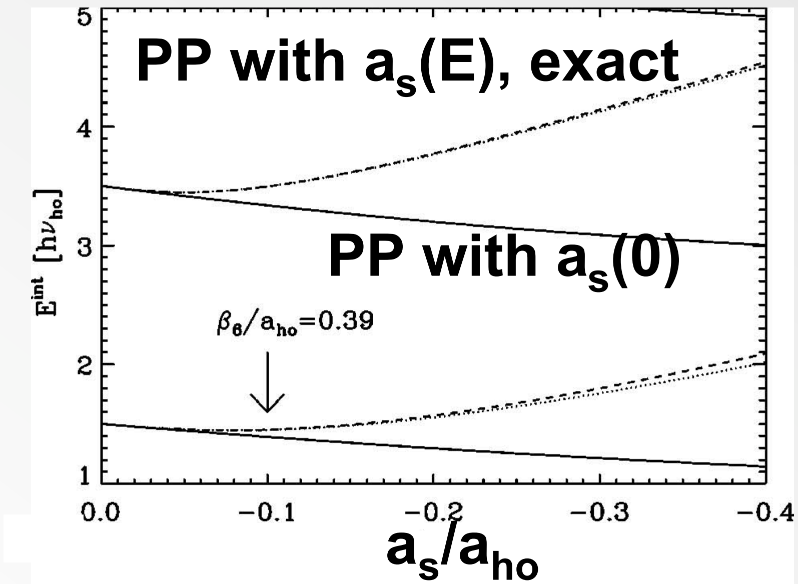
$$V(\vec{r}) = \frac{4\pi\hbar^2 a_s}{m} \delta^{(3)}(\vec{r}) \frac{\partial}{\partial r} r$$

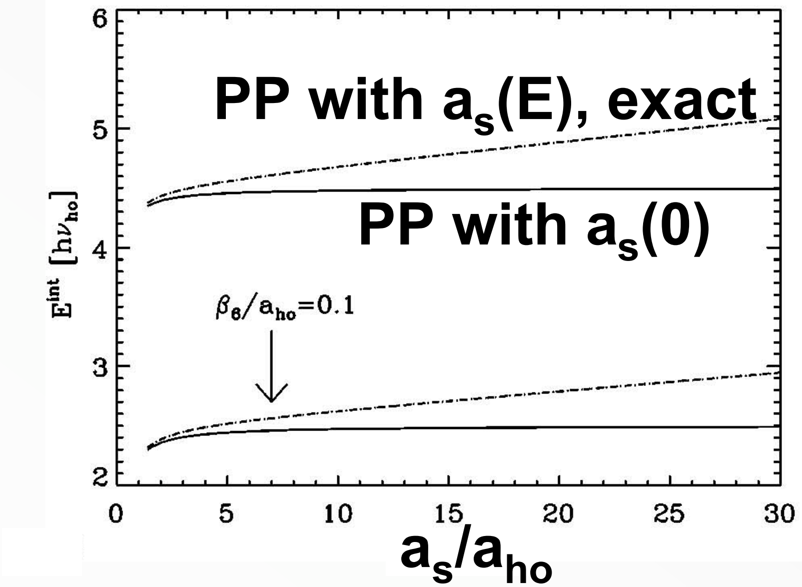
Cures 1/r divergence of radial function (specific to 3D)

Analytical treatments

[Huang and Yang, PR 105, 767 (1957)]

1. Two Particles under External Harmonic Confinement





- Two-particle energy spectrum known semi-analytically: Simple transcendental equation [Busch et al., Found. of Physics (1998)].
- Self-consistent solution when $a_s=a_s(E)$ [Blume and Greene, PRA 65,043613 (2002); see also Bolda et al. (PRA, 2002)].
- Energy-independent pseudo-potential, I.e., use of $a_s(0)$, works if $|a_s| \ll a_{ho}$.
- Energy-dependent pseudo-potential, I.e., use of $a_s(E)$, works if $r_{vdW} \ll a_{ho}$.

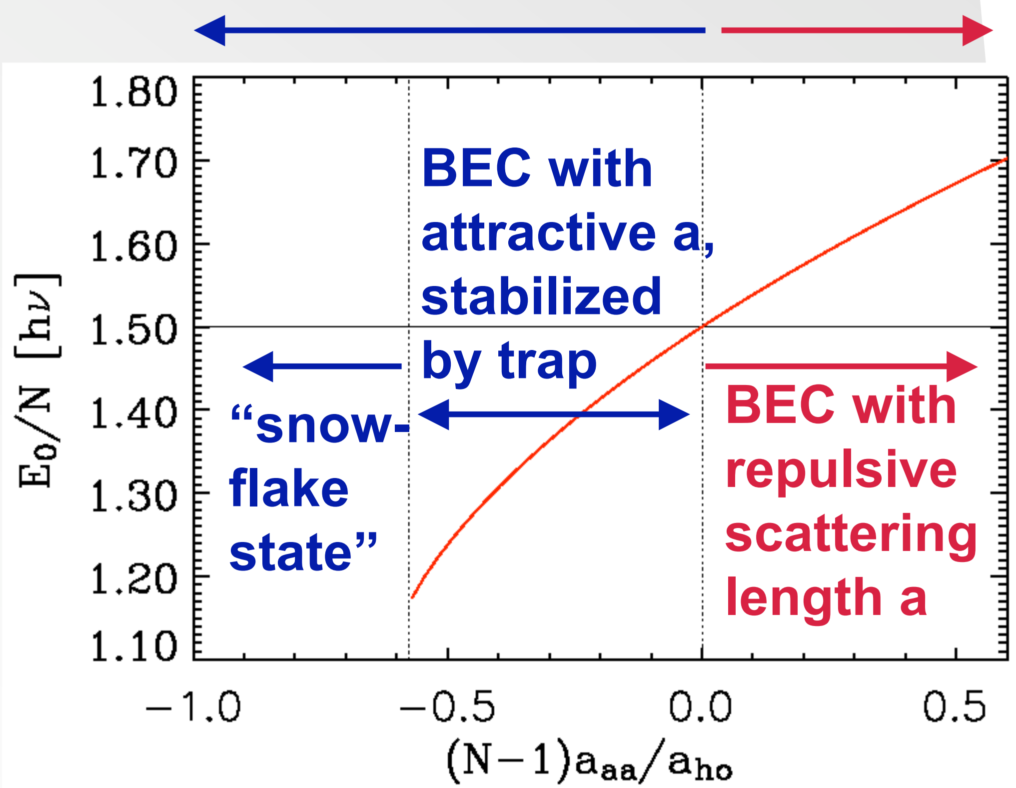
2. Stability of Bose Gas Under Harmonic Confinement

N bosons in a box. Constant density. Periodic boundary conditions.

Positive scattering length: Stable gas (not self-bound).
Negative scattering length: Gas not stable; collapse toward solid or liquid (self-bound).

Bosonic atoms in harmonic trap (mean-field GP treatment):

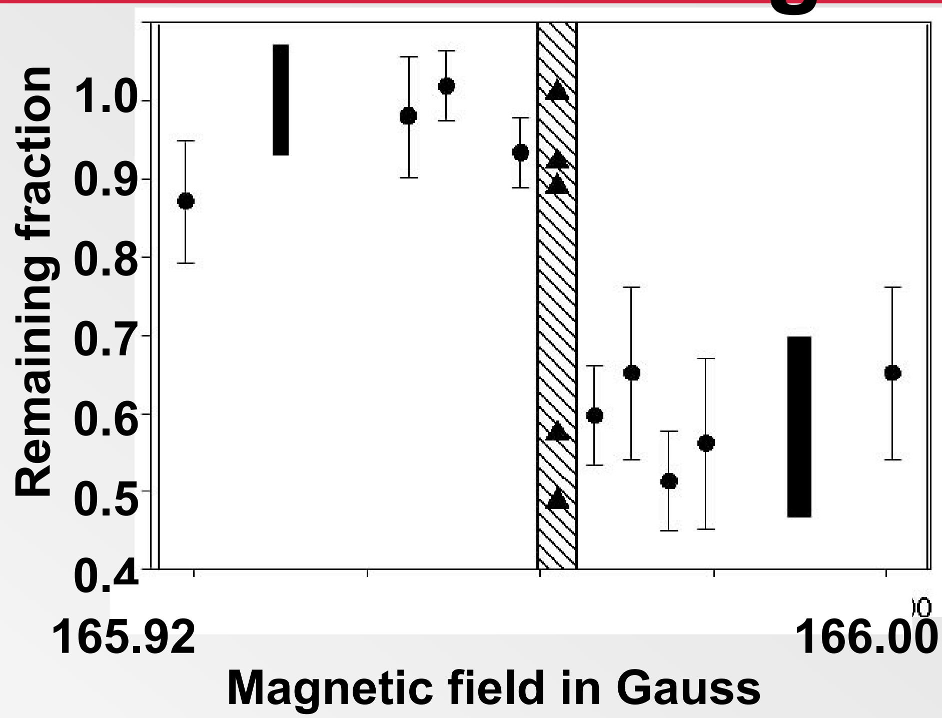
Positive scattering length: Effectively repulsive interaction.
Negative scattering length: Effectively attractive interaction.



Dodd et al., PRA 54, 661 (1996)

2. Experiments with Bosonic ^{85}Rb: Probing Critical Mean-Field Strength

Roberts et al., PRL 86, 4211 (2001)



- **Mean-field prediction confirmed by experiment.**
- **Underlying physical mechanism? Three-body recombination.**

- **Physical picture? Connection between MF GP eq. and many-body Hamiltonian?**

→ more negative a_s

↑ $(N-1)a_s/a_{ho} \sim -0.58$

2. Mean-Field Equation Derived from Many-Body Hartree Wave Function

- **Many-body Hamiltonian for N bosons under confinement:**

$$H = \sum_{j=1}^{N} \left[\frac{-\hbar^2}{2m} \nabla^2_{\vec{r}_j} + \frac{1}{2} m\omega^2 \vec{r}_j^{\,2} \right] + \sum_{j<k}^{N} V_{aa}(\vec{r}_j - \vec{r}_k)$$

SW, HS,…

- **Hartree product (restricted Hilbert space):**

$$\psi(\vec{r}_1, \cdots, \vec{r}_N) = \prod_{i=1}^{N} \phi_a(\vec{r}_i)$$

- **ZR atom-atom potential:**

$$V_{aa}(\vec{r}) = \frac{4\pi\hbar^2 a_{aa}}{m} \delta^{(3)}(\vec{r}) \propto U_{aa} \delta^{(3)}(\vec{r})$$

- **Gross-Pitaevskii (GP) equation for "single atom":**

$$\left(-\frac{1}{2} \nabla^2_{\vec{r}} + \frac{1}{2} \vec{r}^{\,2} + U_{aa}(N-1)|\phi_a(\vec{r})|^2 \right) \phi_a(\vec{r}) = \epsilon_a \phi_a(\vec{r})$$

Single atom feels effective potential/mean-field created by the other N-1 atoms.

2. Mechanical Instability or Collapse of Trapped Bose Gas

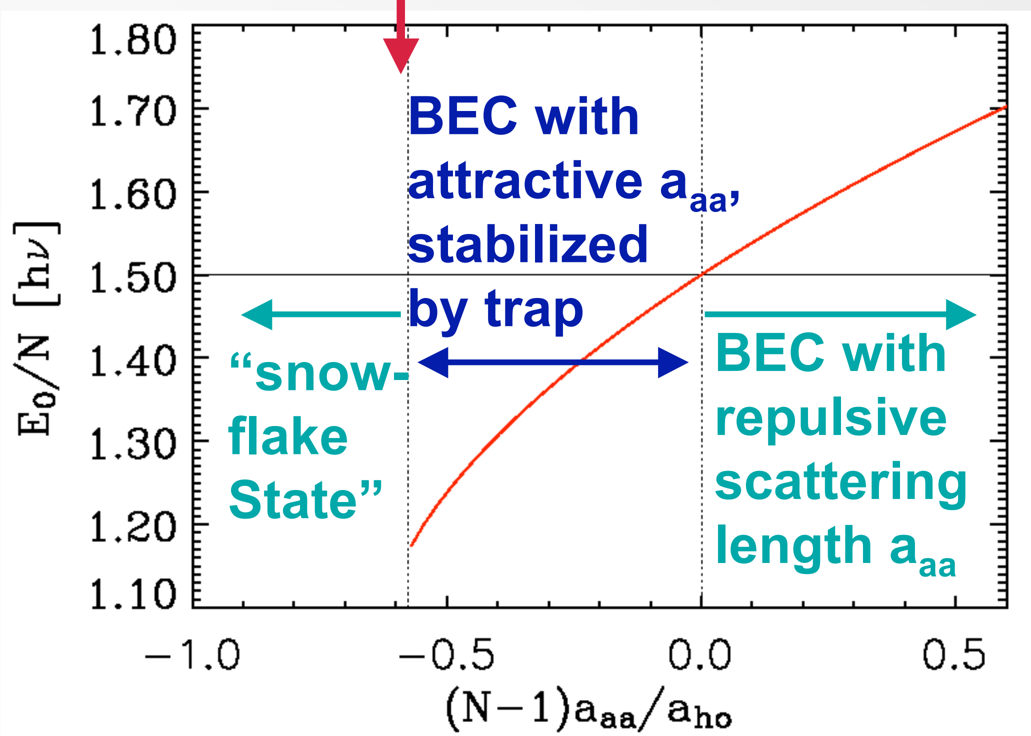
$$\left(-\frac{1}{2}\nabla_{\vec{r}}^2 + \frac{1}{2}\vec{r}^2 + U_{aa}(N-1)|\phi_a(\vec{r})|^2\right)\phi_a(\vec{r}) = \epsilon_a\phi_a(\vec{r})$$

1 parameter

Single particle orbital

Non-linearity

No pos. E solution to GP eq.



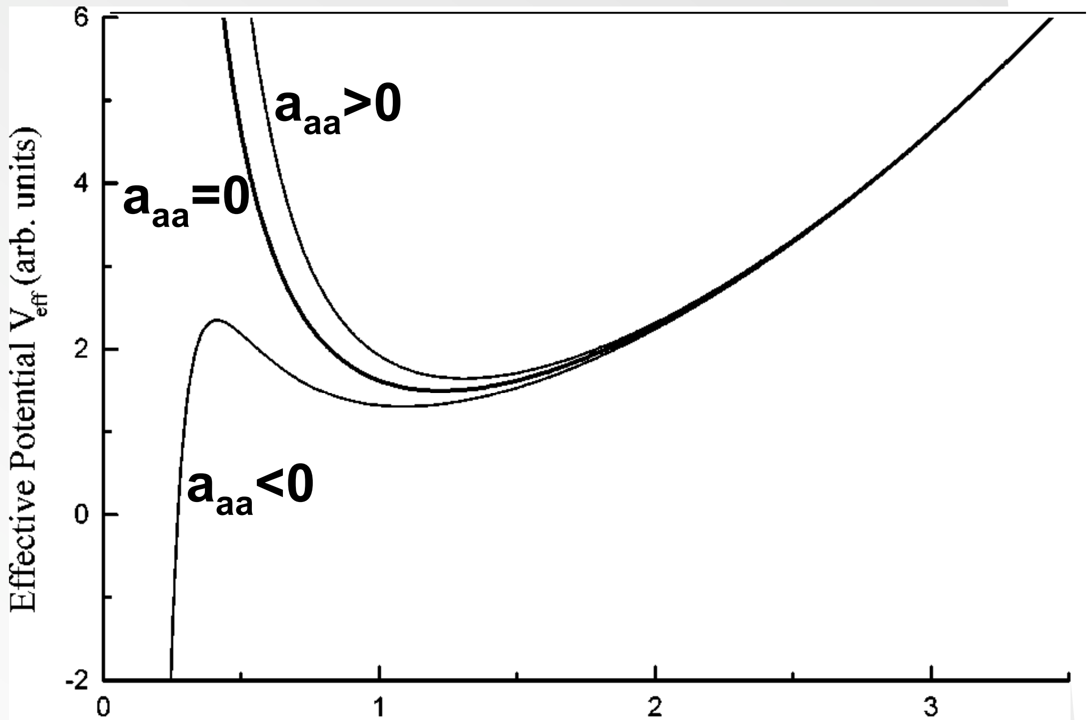
Why does ZR potential work?

At collapse point, $n(0)|a_{aa}|^3 \ll 1$ (small parameter; long wave length approximation justified).

2. Interpretation within Hyperspherical Framework: Linear Schroedinger Eq.

- Coordinates: 1 hyperradius R, (3N-4) hyperangles Ω.
- Many-body symmetrized variational wave function: $F(R)\Phi(\Omega)$.
- Sum of two-body delta-function interactions.
- Effective potential: $V_{eff}(R) = c_1/R^2 + c_2R^2 + c_3a_{aa}/R^3$

Collapse prediction within ~20% of GP equation and experiment.



Hyperradius R in oscillator units

Bohn, Esry and Greene, PRA 58, 584 (1998)

2. Hyperspherical Coordinates: Three Step Approach

$$M = Nm$$

$$\Psi_{NI}(\vec{r}_1, \ldots, \vec{r}_N) = G(\vec{R}_{CM})F(R)\Phi(\vec{\Omega})$$

$$R^2 = \frac{1}{N}\sum_{i=1}^{N}(\vec{r}_i - \vec{R}_{CM})^2$$

$$H_{NI} = H_{CM} - \frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial R^2} + \frac{3N-4}{R}\frac{\partial}{\partial R}\right) + \frac{\Lambda^2}{2MR^2} + \frac{1}{2}M\omega^2R^2$$

$$H_{CM} = \frac{-\hbar^2}{2M}\vec{\nabla}^2_{CM} + \frac{1}{2}M\omega^2R^2_{CM}$$

$$E_{CM} = (2n_{CM} + l_{CM} + 3/2)\hbar\omega$$

1: Remove CM degrees of freedom.

2: Solve hyperangular equation (gives $V_{eff}(R)$).

$$\Lambda^2\Phi_{\lambda,\mu}(\vec{\Omega}) = \hbar^2\lambda(\lambda + 3N-5)\Phi_{\lambda,\mu}(\vec{\Omega})$$

3: Solve hyperradial equation.

With interactions: delta-function interactions can be treated analytically for lowest hyperspherical harmonic.

3. Similarly: Analyze Stability of Multi-Component Fermi Gases

- Simple variational wave function (ideal gas nodal surface) applied to multi-component Fermi gas with “bare” zero-range interactions (all s-wave scattering lengths equal) predicts collapse at:
  - 2-component gas: $(k_F a_{aa})^3 \sim -1.81$ (not small compared to 1)
  - 3-component gas: $(k_F a_{aa})^3 \sim -0.23$
  - 4-component gas: $(k_F a_{aa})^3 \sim -0.067$ (small compared to 1)
- However: Two-component Fermi gas is found to be stable (experiment plus Monte Carlo).

↓ How many components are needed to get instability? Proper theoretical framework?

- Multi-component Fermi gas with one atom per component (Bose gas): Unstable at unitarity.

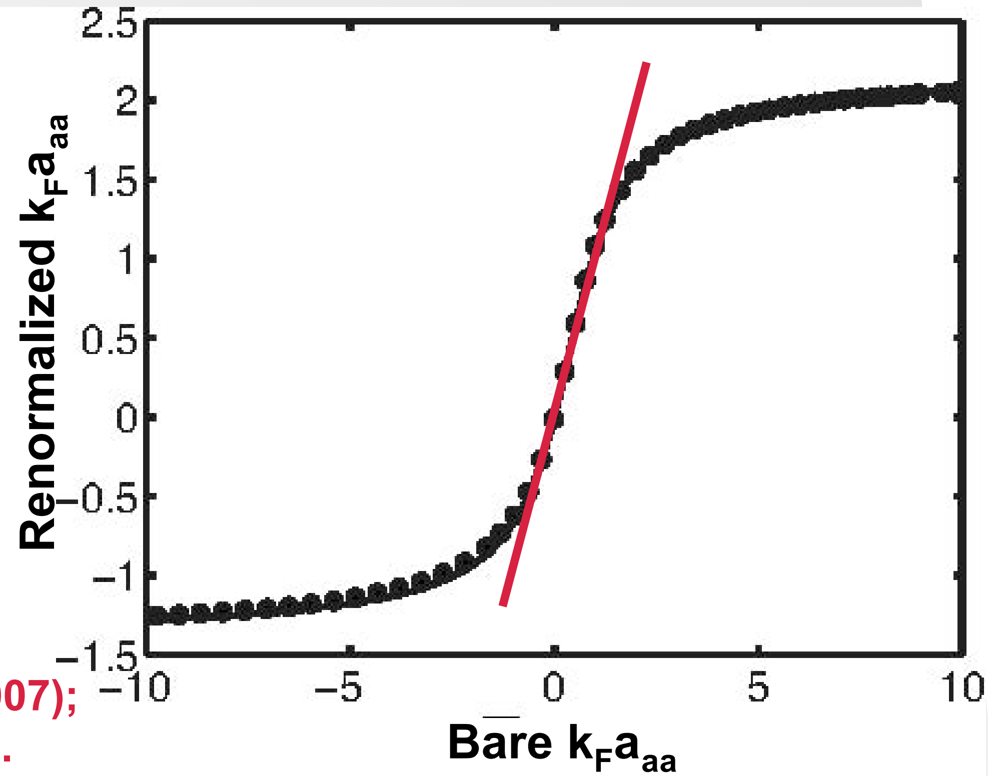
Blume, Rittenhouse, von Stecher, Greene, PRA 77, 032703 (2008).

3. Two-Component Fermi Gas: Renormalization of Interaction

- Many-body anti-symmetrized wave function: $F(R)\Phi(\Omega)$.
- Effective potential: $V_{eff}(R) = c_1/R^2 + c_2R^2 + c_3k_Fa_{aa}/R^3$

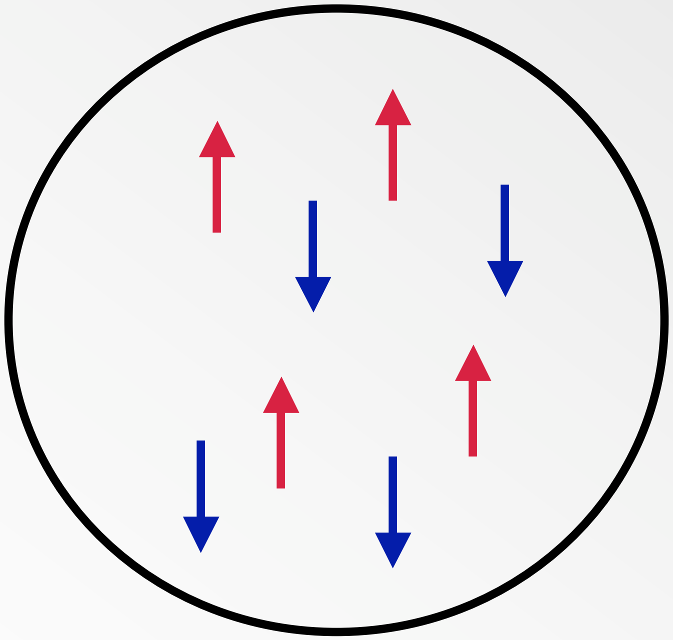
Renormalized ZR potential:
For $|a_{aa}|\rightarrow\infty$: const/$R^2$

At collapse point, $k_F|a_{aa}|$~1.
ZR potential no longer applicable.
Renormalize $k_F|a_{aa}|$ so as to reproduce two-particle spectrum.
Use renormalized $k_F|a_{aa}|$ in hyperspherical framework.

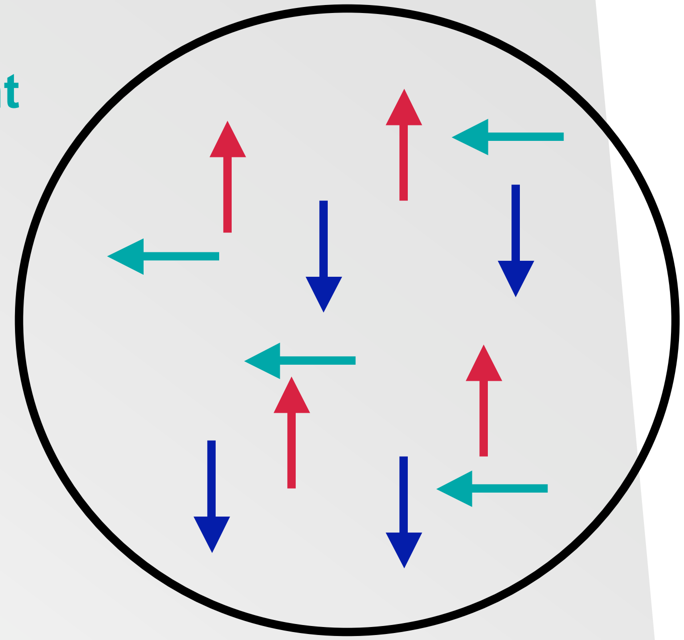


Rittenhouse, Cavagnero, von Stecher and Greene, PRA 74, 053624 (2006);
von Stecher and Greene, PRA 75, 022716 (2007);
Rittenhouse and Greene, physics/0702161v2.

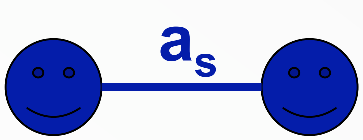
3. Why Collapse for More Components? “Counting Argument”



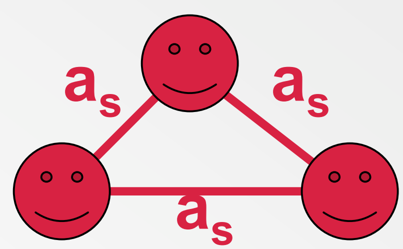
Add a third component
(equal spins:
non-interacting;
unequal spins:
attractive interaction)



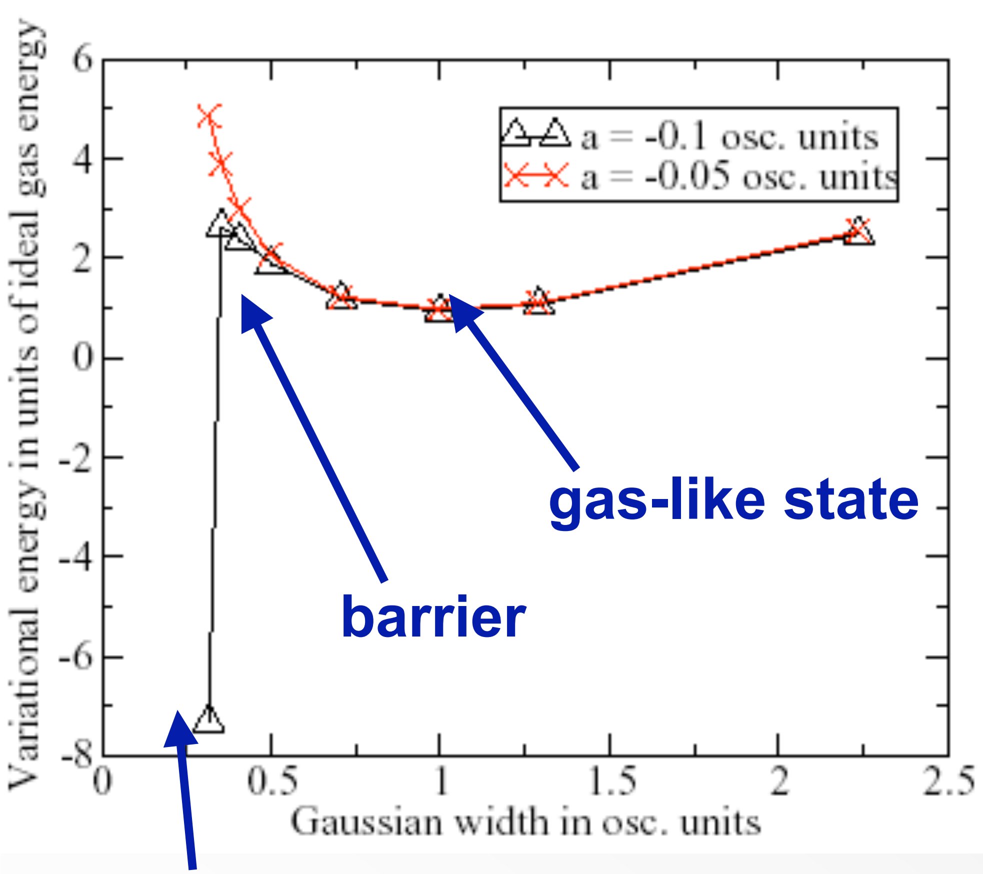
Fermi statistics (repulsion):
12 equal spin pairs.
Attraction:
16 unequal spin pairs.



Fermi statistics (repulsion):
18 equal spin pairs.
Attraction:
48 unequal spin pairs.

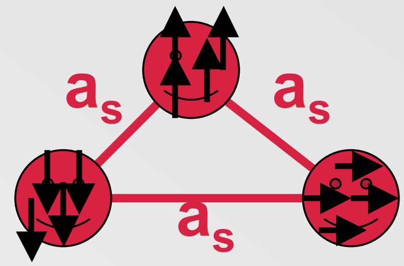


3. Many-Body Variational MC Treatment of Three-Component Fermi System



cluster state (similar to Bose gas with negative scattering length)

- Four fermions per component:



- Square well interaction potential with range $r_0=0.01a_{ho}$.
- Variational wave function with one parameter b that determines size: $\exp(-0.5(r/b)^2)$
- Negative energy state exists if $|a_{aa}|$ "large".
- Peak of barrier at length scale a few times $|a_{aa}|$, $|a_{aa}|<<r_0$.

Blume, Rittenhouse, von Stecher, Greene, PRA 77, 032703 (2008).

Summary and Outlook

- Broad introduction to cold atom physics: Many experiments and strong interactions among people from different communities.

- Revival of few-body physics: Largely due to Efimov physics (fourth lecture).

- Connections between microscopic and macroscopic worlds: Stability of Bose and Fermi gases within many-body framework.

- Next lecture:
  - Few-body techniques and calculations.

General Summary of Field of Cold Atom Physics

- Interaction strengths can be controlled (Fano-Feshbach resonance).
- Confinement can be designed (lattice, quasi-1d,…).

- Fundamental physics question:
  - Strongly-interacting system.
  - Multi-component systems.
  - Equal- and unequal-mass systems.
  - Efimov physics.
  - Dipolar systems.

- Applications:
  - High precision measurements of fundamental constants.
  - Navigational devices.
  - Quantum computation and quantum simulation.