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# Two-dimensional weakly interacting Bose gas: universal equation of state

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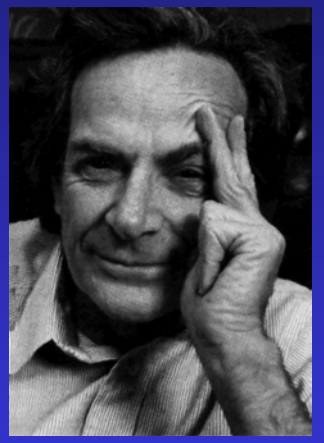
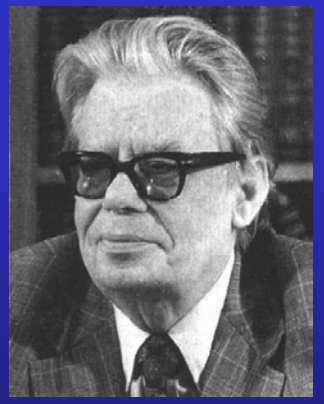
# DILUTE QUANTUM GASES

The study of cold Bose gases historically was an important task as it led to the development of

- perturbative methods
- Feynman diagrams
- Bogoliubov diagonalization approach,
- Beliaev theory
- etc





Many properties at low densities are *universal*.

# UNIVERSAL REGIME

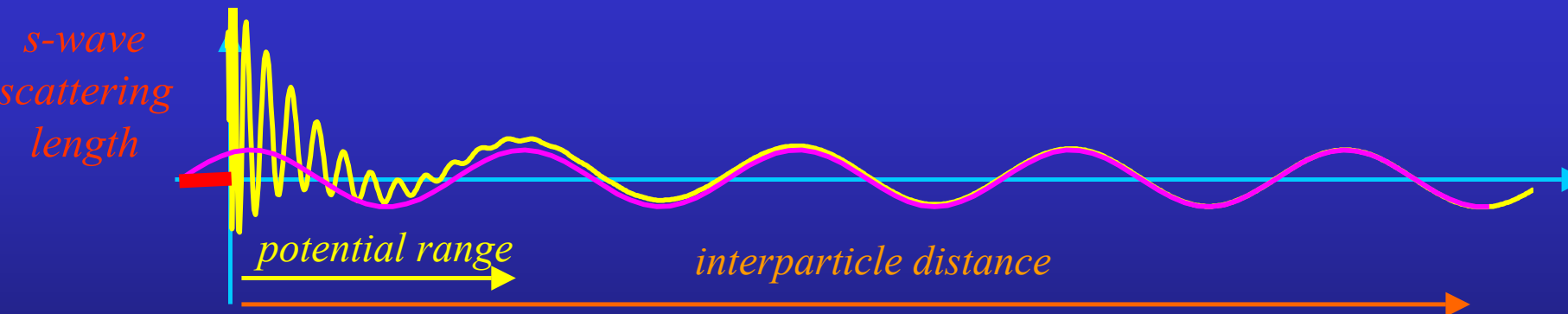
The *universal regime* at ultralow densities can be understood by considering the properties of two-body scattering problem:

$$-\frac{\hbar^2}{2\mu}\Delta f(r) + V_{int}(r) f(r) = E f(r)$$

When the typical interparticle spacing is large compared to the range of the potential, the exact shape of the potential is no longer important and the interaction potential can be describe by a single parameter:

the phase shift or

the *s*-wave scattering length $a_s$




Relevant length scales: $n^{1/3}$ – the interparticle distance and $a_s$.

Properties in the universal regime depend on the *gas parameter* $na^3$

# MEAN-FIELD THEORY

in the mean-field description the
hemical potential is linear with the density: $\mu = gn$

the pseudopotential $V_{pseud}(r) = g\,\delta(r) \times (regularization)$

the coupling constant $g$ being has the dimensionality of

$[g] = [energy \cdot volume] = [energy \cdot L^D]$

Proper units are obtained in the following combinations:

3D: $\left[\frac{\hbar^2 a_{3D}}{m}\right] \rightarrow g_{3D} = 4\pi\frac{\hbar^2 a_{3D}}{m}$

1D: $\left[\frac{\hbar^2}{m a_{1D}}\right] \rightarrow g_{1D} = -2\frac{\hbar^2}{m a_{1D}}$

2D: $\left[\frac{\hbar^2}{m}\right] \rightarrow g_{2D} = 4\frac{\hbar^2}{m}\frac{1}{|\ln n a^2|}$

In 2D:

-dependence on $a_s$ is weak (logarithmic)

- $g_2$ depends on density

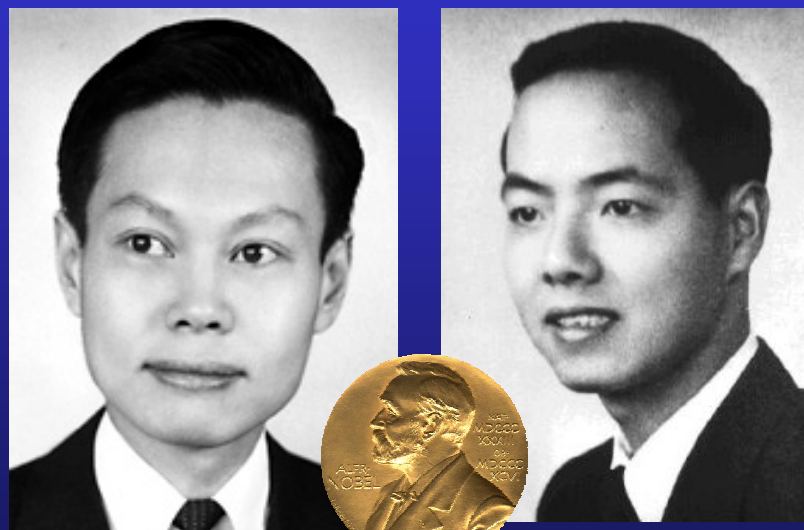
# THREE-DIMENSIONAL SYSTEM

The equation of state of a three-dimensional Bose gas dates back to 1957 when the beyond mean-field correction was obtained by K.W. Huang and Nobel prize winners C. N. Yang and T. D. Lee Huang, Yang, and Lee .

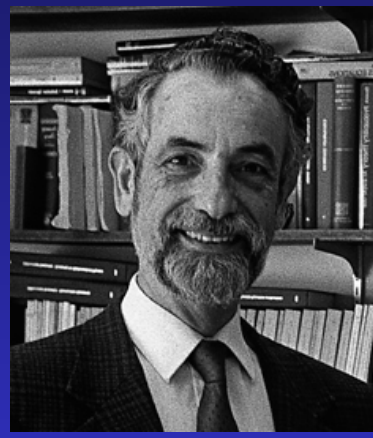
$$\frac{E}{N} = \frac{1}{2} g_{3D} a \left[ 1 + \frac{128}{15\sqrt{\pi}} \sqrt{na^3} + \frac{8(4\pi - 3\sqrt{3})}{3} na^3 \ln na^3 + \ldots \right]$$

-expansion comes in powers of the gas parameter $na^3$

- there is a subleading logarithmic term



# ONE-DIMENSIONAL SYSTEM



The equation of state of a one-dimensional Bose gas was obtained

- for pseudopotential gas by E. H. Lieb and W. Liniger in 1963 using Bethe *ansatz*. An explicit expression for the energy can be written in the weakly interacting limit in terms of $1 / (n_{1D} a_{1D})$ :

$$\frac{E}{N} = \frac{1}{2} g_{1D} n_{1D} \left( 1 - \frac{4\sqrt{2}}{3\pi} (n_{1D} |a_{1D}|)^{-1/2} + \ldots \right)$$

- for hard rod gas by M. Girardeau in 1960 by constructing a Bose-Fermi mapping for the ground state wave function. The expansion comes in terms of $n_{1D} a_{1D}$:

$$\frac{E}{N} = \frac{\pi^2 \hbar^2 n_{1D}^2}{6m} (1 + 2 n_{1D} a_{1D} + \ldots)$$

# TWO-DIMENSIONAL SYSTEM: MF

The mean-field contribution to the equation of state of a two-dimensional Bose gas was obtained by M. Schick in 1971. The chemical potential is given by:

$$\mu = g_{2D} n = \frac{4\pi\hbar^2 n}{m} \frac{1}{|\ln na^2|}$$

The energy is then obtained by integrating the chemical potential over the number of particles:

$$\frac{E^{MF}}{N} = \frac{2\pi n\hbar^2/m}{|\ln na^2| + 1/2 - 1/(4|\ln na^2|) + \ldots} \approx \frac{1}{2} gn$$

- expansion comes in terms of the logarithm of the gas parameters ln *na*
- convergence is very slow, compared to 3D and 1D cases
- calculation of the beyond mean-field terms is very complicated

# NATURE OF BEYOND MEAN-FIELD TERMS

- Quantum fluctuations deplete the condensate and provide a BFM contribution to the energy:

In 3D: $\frac{N_0}{N} = 1 - \frac{8}{3\sqrt{\pi}}\sqrt{na^3} + \ldots$, BMF term $\propto \sqrt{na^3}$

In 2D: $\frac{N_0}{N} = 1 - \frac{1}{|\ln na^2|} + \ldots$, BMF term $\propto \frac{1}{|\ln na^2| + const}$

- integrating MF expression for the chemical potential: $\propto \frac{1}{|\ln na^2| + const + \frac{const}{|\ln na|}}$

- Popov's recursive relation for the chemical potential: $\propto \frac{1}{|\ln na^2| + \ln|\ln na^2| + 4\pi \times const + \frac{const}{|\ln na^2}}$

-the difference between cut-off length, range of the potential and $a$.

- chemical potential depends on the scattering amplitude (in the ladder approximation), which contains $\gamma$ in its short-range expansion

# TWO-DIMENSIONAL SYSTEM: BMF

| year | first author | type | terms |
|---|---|---|---|
| 1971 | Schick | MF | $E^{MF}/N = 2\pi\hbar^2 n/(m\lvert\ln na^2\rvert)$ |
| 1971 | Popov | | $\ln\lvert\ln na^2\rvert - \ln 4\pi - 1/2$ |
| 1978 | Lozo | | $\ln\lvert\ln na^2\rvert - \ln 4\pi + 1/2$ |
| 1978 | H | | $\ln\lvert\ln(\ldots^2/\pi)\rvert - \ln 2\pi^3 - 2\gamma + 3/2$ |
| 1988 | Fi | | $\ln\lvert\ln\ldots^2\rvert - \ln 4\pi - 1/2$ |
| 1992 | Ko | | $\ln\lvert\ln\ldots na^2)\rvert - \ln 4\pi$ |
| 1993 | Ov | | $\ln\lvert$ |
| 2001 | Li | | $\ldots = 2\pi\hbar^2 n/(m\lvert\ln na^2\rvert)$ |
| 2001 | C | | $\ln\lvert\ldots^2\rvert - \ln\pi - 2\gamma - 1/2$ |
| 2002 | And | | $\ln\lvert\ln na^2\rvert - \ln 4\pi - 1/2$ |
| 2003 | Mora | | $\ln\lvert\ln na^2\rvert - \ln\pi - 2\gamma - 1/2$ |
| 2003 | Pricoupenko | BMF | $\ln\lvert\ln na^2\rvert - \ln\pi - 2\gamma - 1/2$ |
| 2005 | Pilati | BMF | $0.86\ln\lvert\ln na^2\rvert - 2.26$ |

**Ingredients are there:**
**terms contain:**
**1, π, γ**
**But what is the correct combination???**

# HOW TO SEE BEYOND MEAN-FIELD TERMS

Analytic studies agree that the first BMF term is a double logarithm $C_1 = 1$, while there is no agreement on the second BMF term $C_2$

$$\frac{E}{N} = \frac{2\pi n\hbar^2/m}{|\ln na^2| + C_1 \ln|\ln na^2| + C_2 ...}$$

Typical density of applicability of the $1^{st}$ term: the term $\ln|\ln na$ be $m$ times larger than typical constant term $\ln 4\pi$. This leads to a estimation of a characteristic density $na^2 \approx e^{-(4\pi)^m}$.

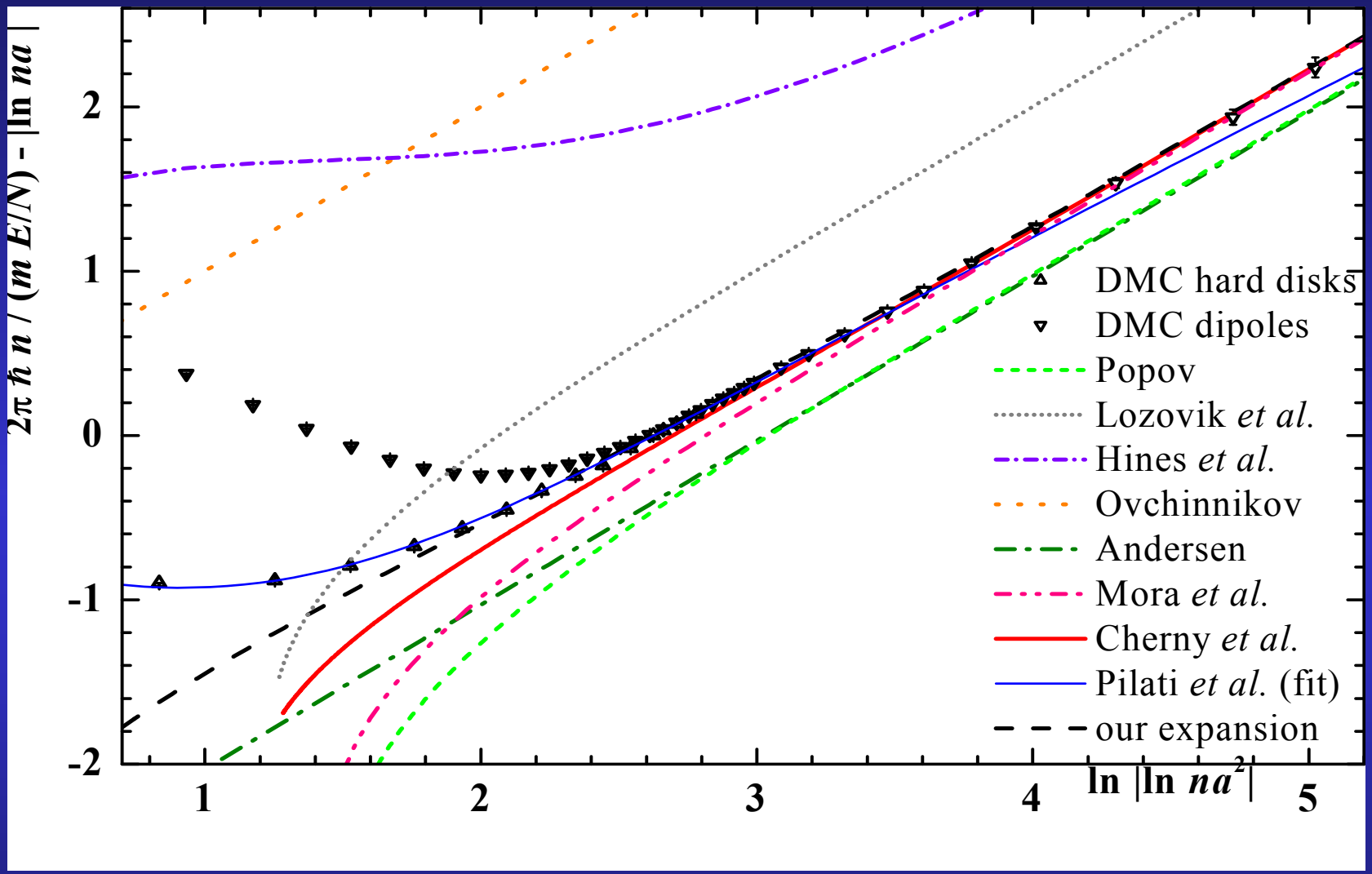
- 50% accuracy $(m = 2) \rightarrow$ extremely low density $na^2 \approx 10^{-69}$
- 33% accuracy $(m = 3) \rightarrow$ unrealistic number $na^2 \approx 10^{-862}$.

Beyond mean-field terms can be conveniently seen in the expressio

$$\frac{2\pi n\hbar^2/m}{E/N} - |\ln na^2| = C_1 \ln|\ln na^2| + C_2 ...$$

in the limit of small density, $\ln|\ln na^2| \gg 1$, as a straight line.

# RESULTS: BEYOND MEAN-FIELD ENERGY




Non-universal beyond MF corrections $2\pi\hbar^2 n/[mE/N] - |\ln na^2|$ in the energy per particle as a function of $\ln|\ln na^2|$ . Symbols, DMC results: up triangles, hard disks, down triangles dipoles

# BETTER EXPANSION PARAMETER

MC results agree within errorbars with the constant

$$C_2 = -\ln \pi - 1/2 - 2\gamma,$$

or $\ln|\ln na^2| \gtrsim 3$. This term was analytically obtained by

A. Cherny& A. Shanenko (2001),
C. Mora&Y. Castin (2003),
L. Pricoupenko (2004).

Cherny and Shanenko do a clever expansion in terms of a imensionless in-medium scattering amplitude $u$ obeying the equatio

$$1/u + \ln u = -\ln(\pi n a^2) - 2\gamma.$$

mportantly, defined in this way, $u$ contains all recursive terms of ype $|\ln na^2| + \ln|\ln na^2| + \ln\ln|\ln na^2| + ...$

The dimensionless energy $\varepsilon = Em/(2\pi\hbar^2 nN)$ is then expanded s a series $\varepsilon = u + u^2/2 - u^3 + ...$ [Cherny & Shanenko].

# FIT OF THE UNIVERSAL EQUATION OF STATE

We test the accuracy of the series $\varepsilon = u + u^2/2 - u^3 + \ldots$ [Chern
Shanenko] and find that the agreement with the first two term
this expansion is notably good. We apply a fitting procedure t
nd the coefficient in front of $u^3$ with the $\chi^2$ criterion; the result
$E/(2\pi\hbar^2 Nn) = u + u^2/2 - 2.0(1)u^3$. In terms of the gas paramete
ur best perturbative description for the chemical potential is

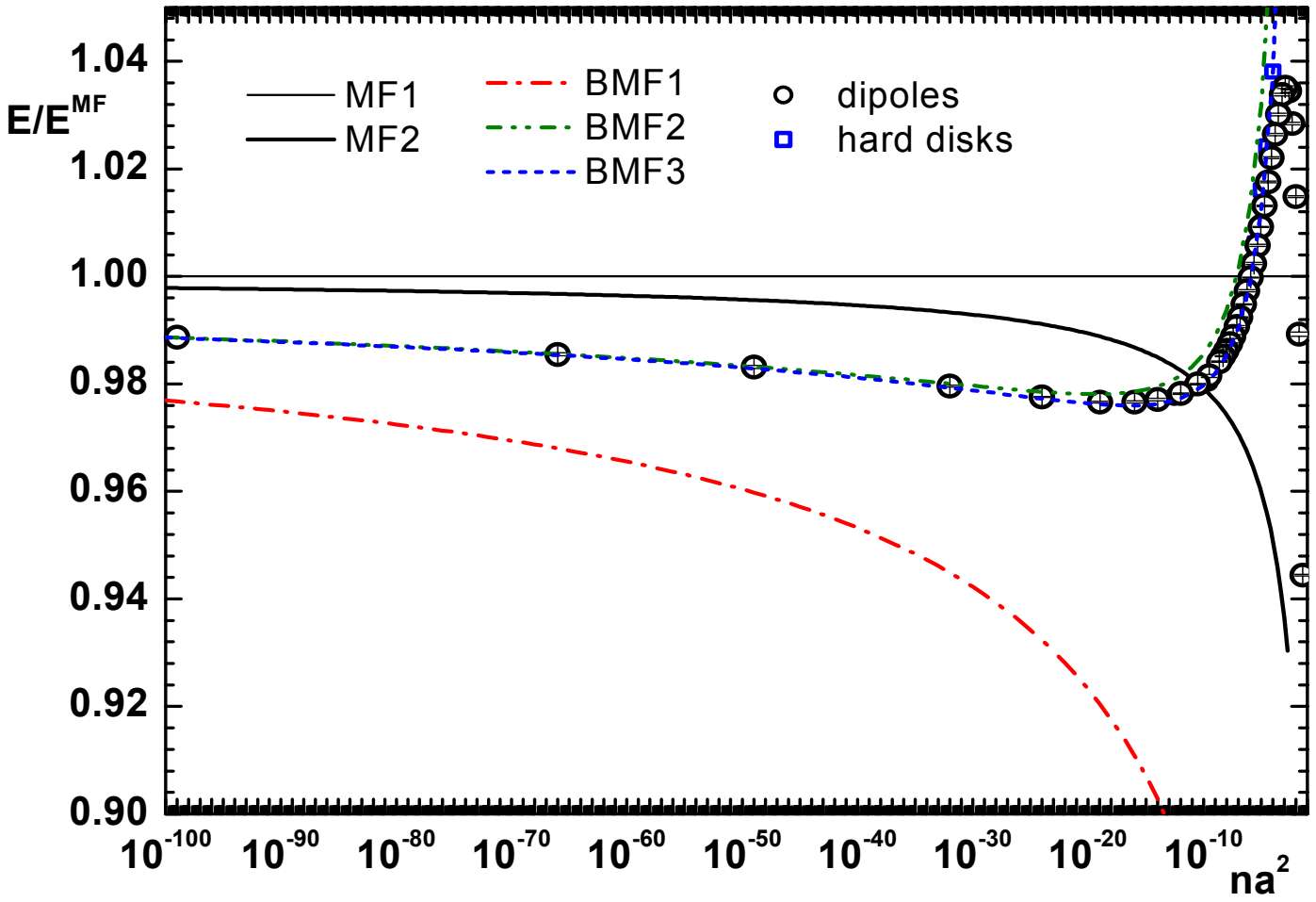
$$\mu = \frac{4\pi\hbar^2 n/m}{|\ln na^2| + \ln|\ln na^2| + C_1^\mu + \frac{\ln|\ln na^2| + C_2^\mu}{|\ln na^2|} + \ldots},$$

ith $C_1^\mu = -\ln\pi - 2\gamma - 1 = -3.30\ldots$ and $C_2^\mu = -\ln\pi - 2\gamma + 2.0(1) = -0.3(1$
he expression for the "universal" energy per particle is

$$\frac{E}{N} = \frac{2\pi\hbar^2 n/m}{|\ln na^2| + \ln|\ln na^2| + C_1^E + \frac{\ln|\ln na^2| + C_2^E}{|\ln na^2|} + \ldots}$$

where $C_1^E = C_1^\mu + 1/2 = -2.80\ldots$ and $C_2^E = C_2^\mu + 1/4 = -0.05(10$

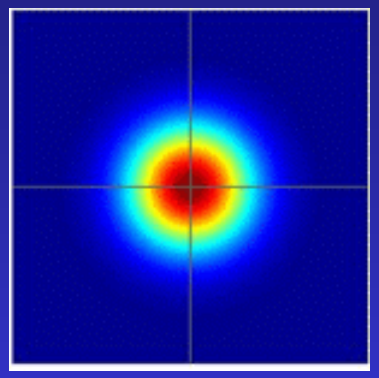
# RESULTS: SERIES CONVERGENCE

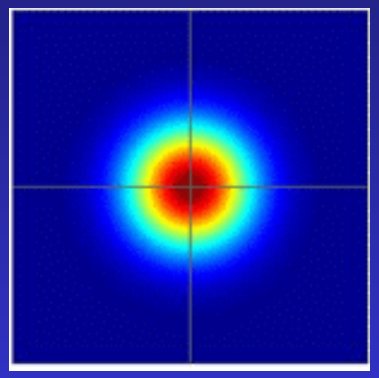



Energy in units of $E_1^{MF}/N = 2\pi\hbar^2 n/|m \ln na^2|$ as a function of $na^2$.
MF1: Schick's energy $E_1^{MF}$, MF2: $E_2^{MF}/N = 4\pi n\Gamma(0, 2|\ln na^2|)\hbar^2/m$

# TRAPPED SYSTEMS

he equation of state can be studied in high precision experiment
easuring the frequency $\Omega$ of the lowest breathing mode in a trap.



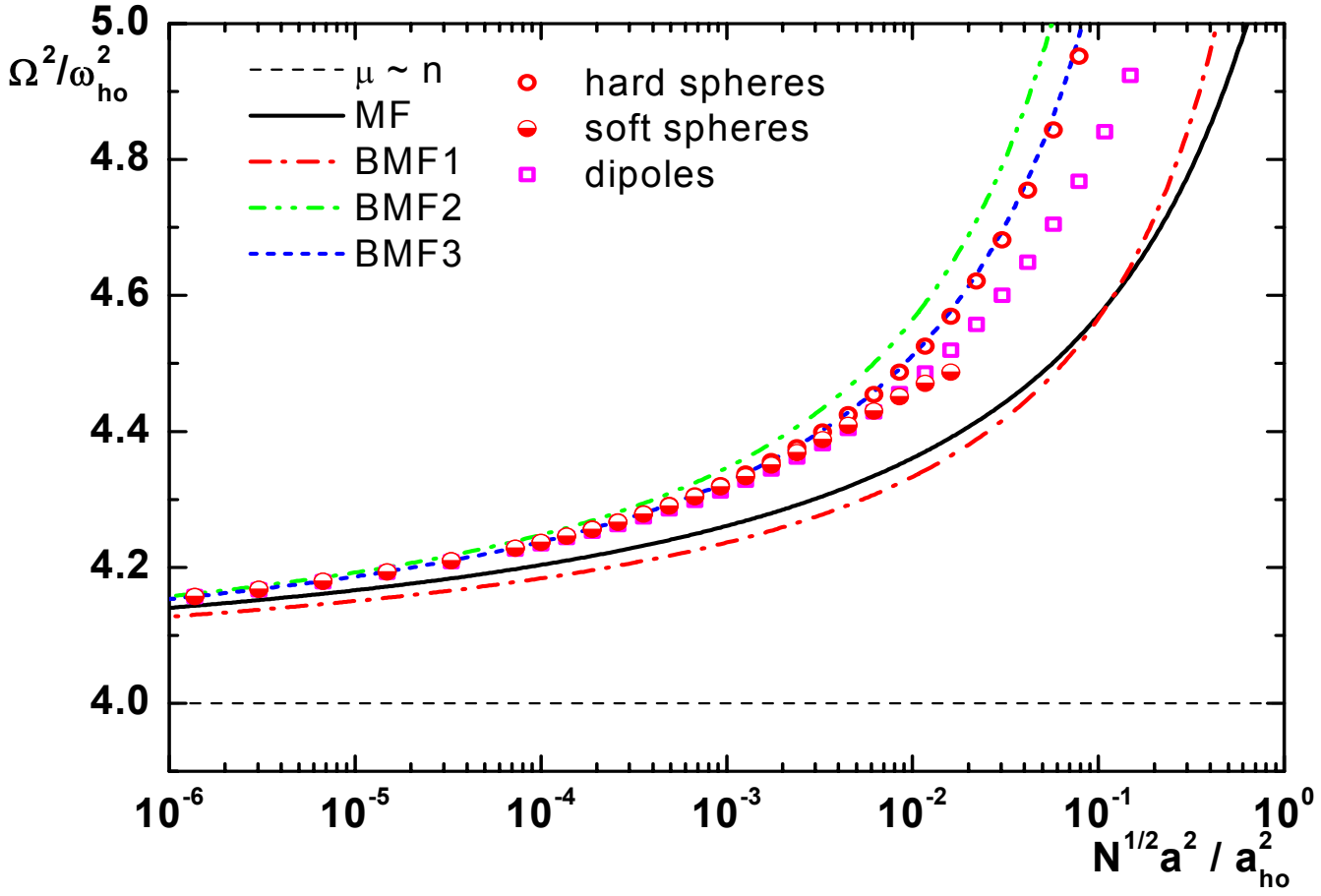


ropertiers of a trapped gas can be described within the local densit
pproximation once the density dependence of the chemical potentia
a homogeneous system $\mu_{hom}(n)$ is known

$$\mu = \mu_{hom}[n(\mathbf{r})] + \frac{1}{2}m\omega_x x^2 + \frac{1}{2}m\omega_y y^2 + \frac{1}{2}m\omega_z z^2$$

onstant $\mu$ is fixed by the normalization condition $N = \int n(\mathbf{r})dr$. Th
equency of the breathing mode is obtained as $\Omega^2_{sph} = -2\langle r^2 \rangle \Big/ \frac{\partial \langle r^2 \rangle}{\partial \omega^2}$
ote that density at the edge of the trap alway vanishes!

# RESULTS: COLLECTIVE OSCILLATIONS




Square of the lowest breathing mode frequency, $\Omega^2$ as a function of the LDA parameter $N^{1/2} r_0^2 / a_{ho}^2$. Lines: simple MF $\mu_0^{MF} \propto n$, Schick's expression, BMF terms; Symbols: DMC results

# CONCLUSIONS

- Universal equation of state of a two-dimensional dilute Bose gas
s a long standing problem. A large number of analytical results is
ound in the literature (often incomplete and even contradictory!)

- All theories agree at mean-field level and the majority agree on
he first beyond MF term, which, however, does not provide an
ccurate quantitative description at realistically low densities.

- This problem is addressed at zero temperature using diffusion
lonte Carlo method. Densities as low as $na^3 = 10^{-100}$ are considere

- Analytical expressions for the beyond mean-field terms are teste
nd confronted to the numerical results in the ultradilute regime

- An empirical equation of state is provided as a fit to DMC data

- we suggest future experimental work to measure the frequency
f a breathing collective oscillation in a trap as a very sensitive too
or verifying the universal 2D equation of state.

THANK YOU
FOR
YOUR ATTENTION!